



**Institute of
Applied Physics**

Friedrich-Schiller-Universität Jena

Imaging and Aberration Theory

Lecture 2: Fourier- and Hamiltonian Optics

2018-10-26

Herbert Gross



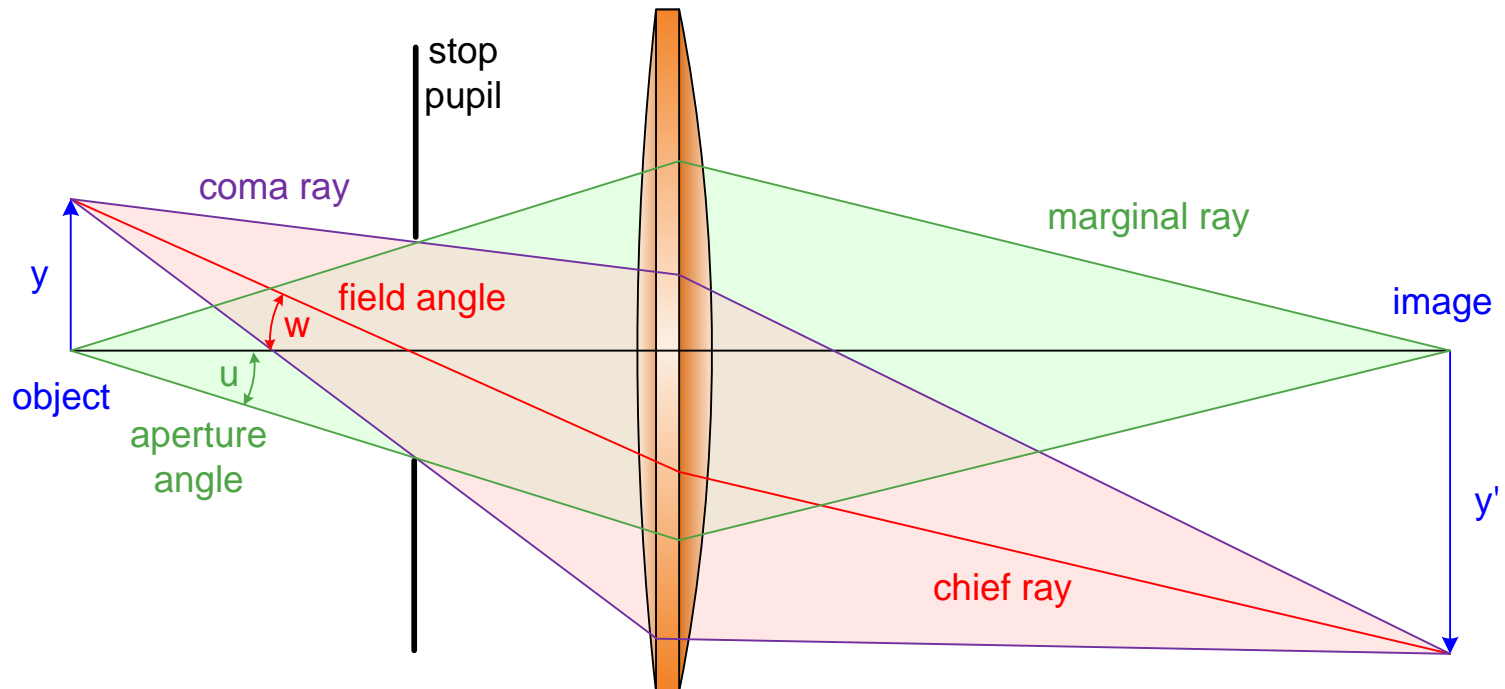
Schedule - Imaging and aberration theory 2018

1	19.10.	Paraxial imaging	paraxial optics, fundamental laws of geometrical imaging, compound systems
2	26.10.	Pupils, Fourier optics, Hamiltonian coordinates	pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates
3	02.11.	Eikonal	Fermat principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media
4	09.11.	Aberration expansions	single surface, general Taylor expansion, representations, various orders, stop shift formulas
5	16.11.	Representation of aberrations	different types of representations, fields of application, limitations and pitfalls, measurement of aberrations
6	23.11.	Spherical aberration	phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders
7	30.11.	Distortion and coma	phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options
8	07.12.	Astigmatism and curvature	phenomenology, Coddington equations, Petzval law, correction options
9	14.12.	Chromatical aberrations	Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum
10	21.12.	Sine condition, aplanatism and isoplanatism	Sine condition, isoplanatism, relation to coma and shift invariance, pupil aberrations, Herschel condition, relation to Fourier optics
11	11.01.	Wave aberrations	definition, various expansion forms, propagation of wave aberrations
12	18.01.	Zernike polynomials	special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, measurement
13	25.01.	Point spread function	ideal psf, psf with aberrations, Strehl ratio
14	01.02.	Transfer function	transfer function, resolution and contrast
15	08.02.	Additional topics	Vectorial aberrations, generalized surface contributions, Aldis theorem, intrinsic and induced aberrations, reverbility

1. Definition of aperture and pupil
2. Special ray sets
3. Vignetting
4. Helmholtz-Lagrange invariant
5. Phase space
6. Resolution and uncertainty relation
7. Hamiltonian coordinates
8. Analogy mechanics – optics

Diaphragm in Optical Systems

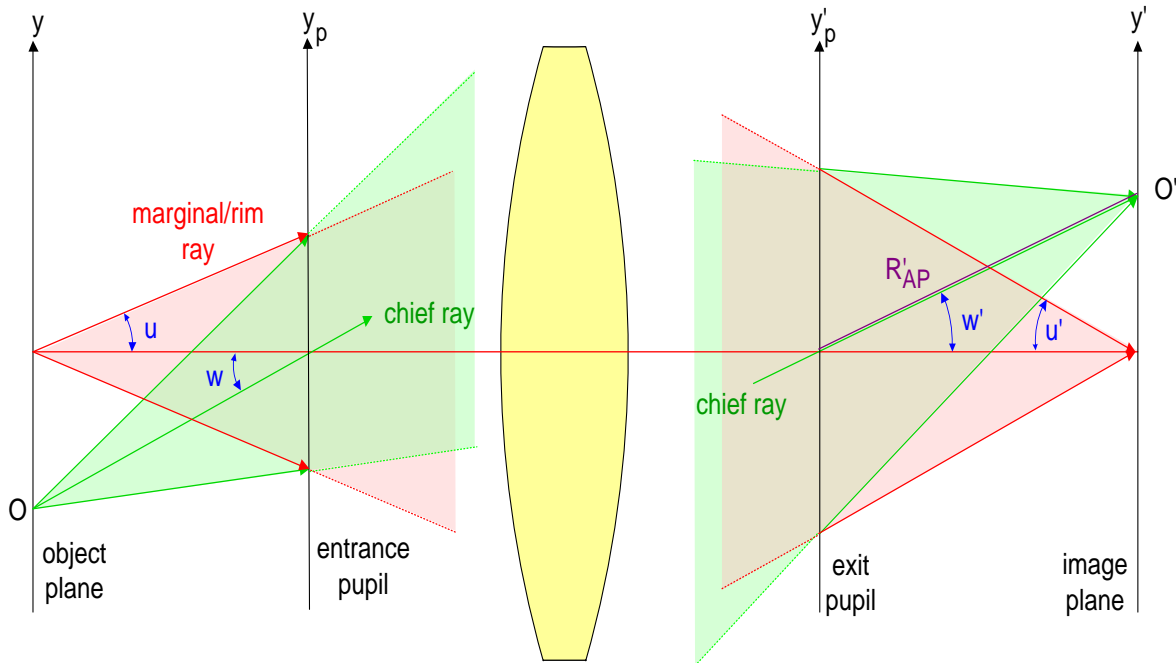
- Pupil stop defines:
 1. chief ray angle w
 2. aperture cone angle u
- The chief ray gives the center line of the oblique ray cone of an off-axis object point
- The coma rays limit the off-axis ray cone
- The marginal rays limit the axial ray cone





Definition of Field of View and Aperture

- Imaging on axis: circular / rotational symmetry
Only spherical aberration and chromatical aberrations
- Finite field size, object point off-axis:
 - chief ray as reference
 - skew ray bundles:
coma and distortion
 - Vignetting, cone of ray bundle
not circular symmetric
 - to distinguish:
tangential and sagittal
plane





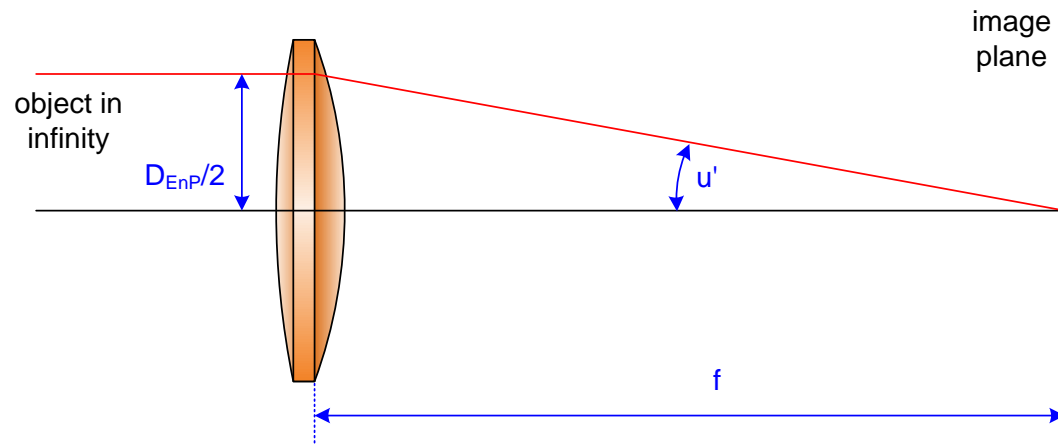
Numerical Aperture and F-number

- Classical measure for the opening: numerical aperture

$$NA' = n \cdot \sin u'$$

- In particular for camera lenses with object at infinity: F-number

$$F_{\#} = \frac{f}{D_{EnP}}$$



- Numerical aperture and F-number are to system properties, they are related to a conjugate object/image location

- Paraxial relation $F_{\#} = \frac{1}{2n' \tan u'}$

- Special case for small angles or sine-condition corrected systems

$$F_{\#} = \frac{1}{2NA'}$$

- More general definition of the F-number for systems with finite object location
- Effective or working F-number with

$$s' = f' \cdot (1 - m)$$

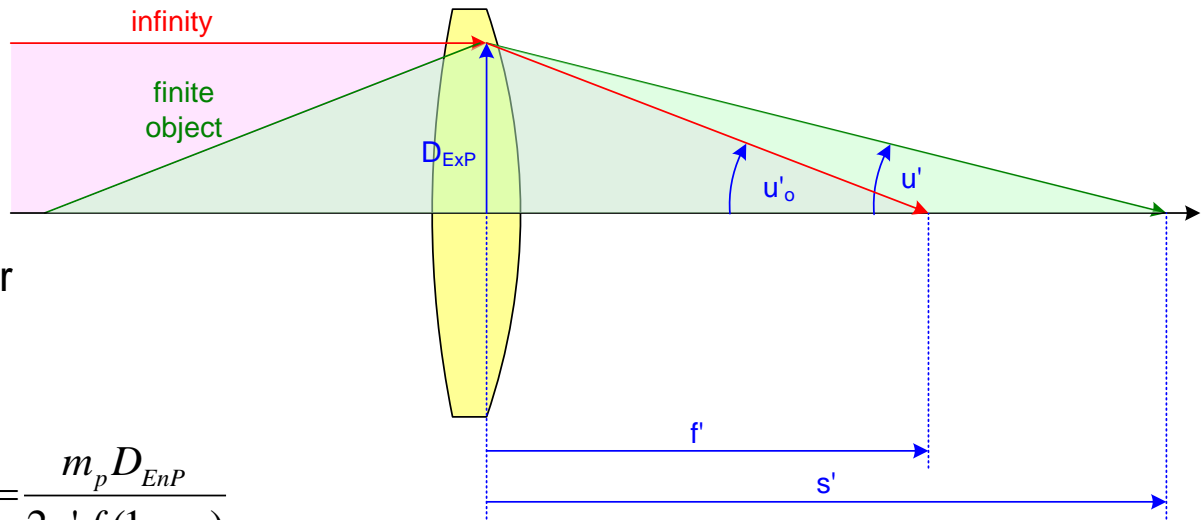
and

$$\sin u' = \frac{D_{Exp}}{2s'} = \frac{D_{Exp}}{2f'(1-m)} = \frac{m_p D_{EnP}}{2n' f(1-m)}$$

we get as a relation with the object-in-infinity-case

$$F_{\#}^{eff} = \frac{1}{2n' \cdot \sin u'} = F_{\#} \cdot \frac{1-m}{m_p}$$

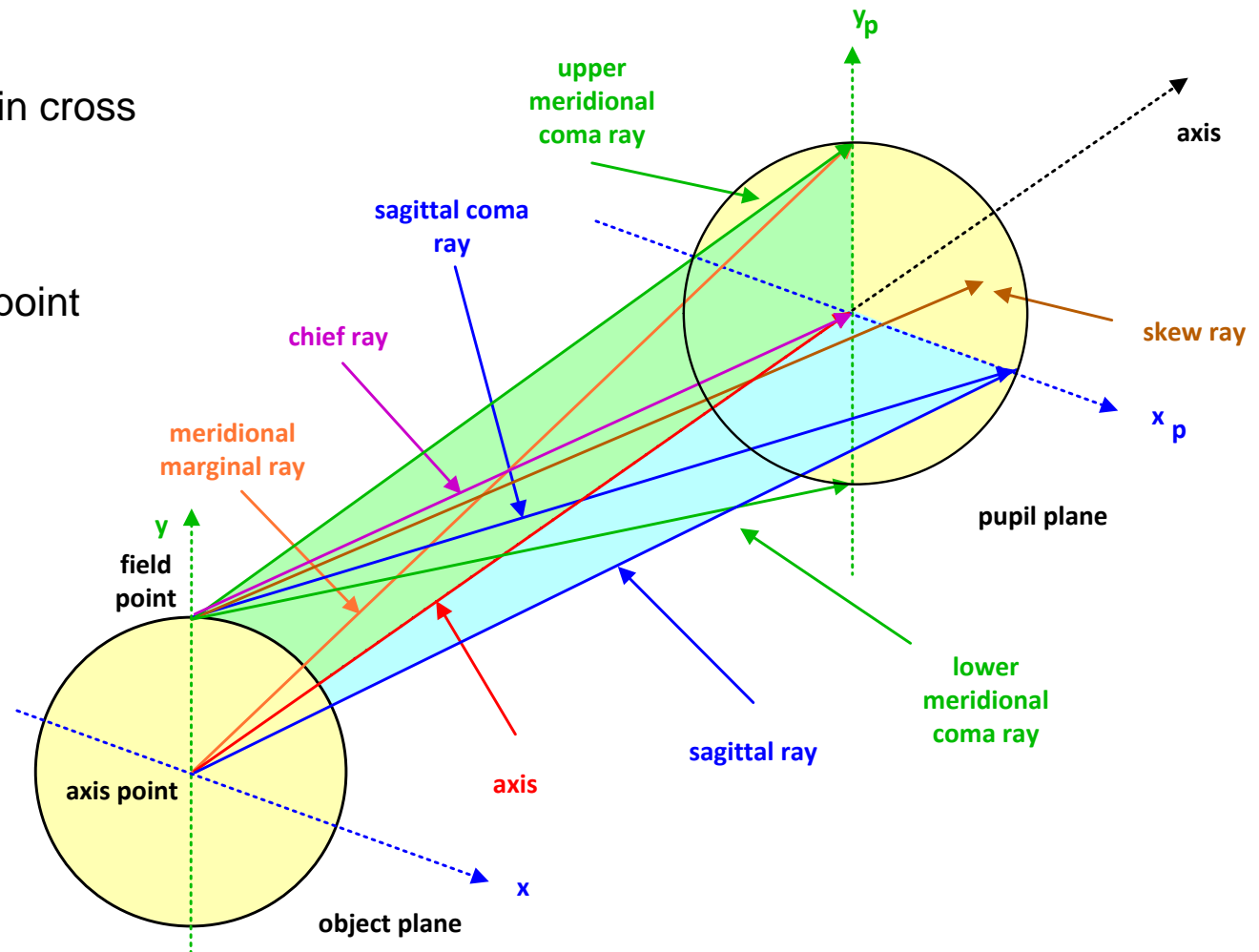
m is the system magnification, m_p is the pupil magnification



Special rays in 3D



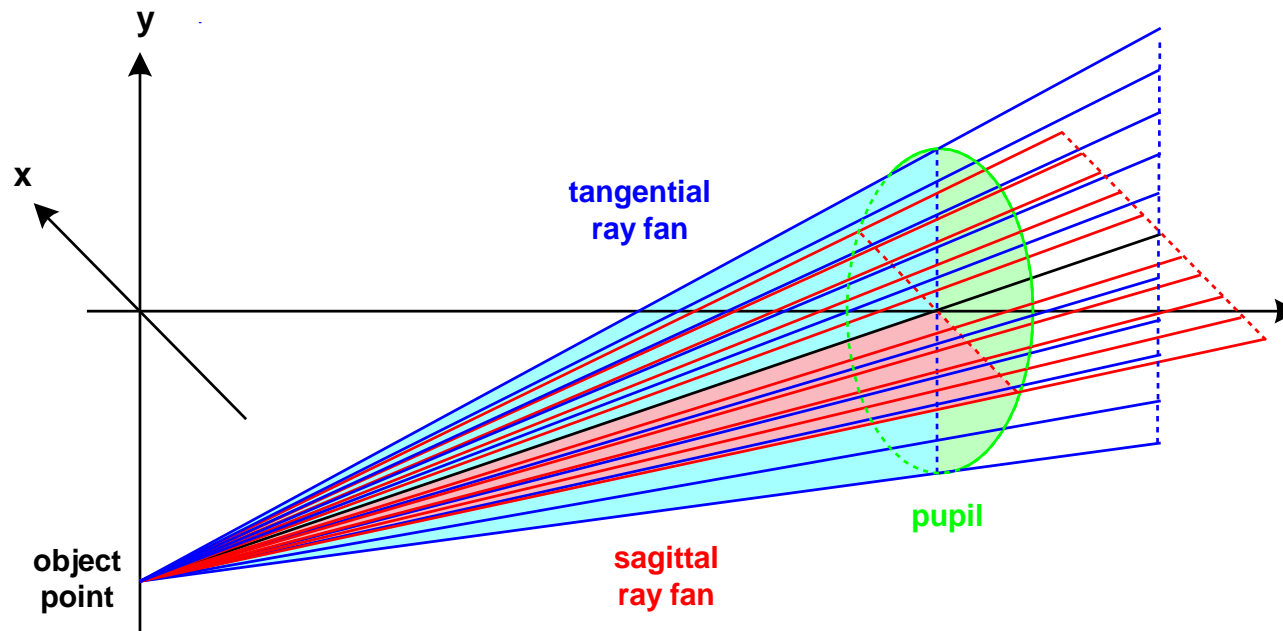
- Meridional rays:
in main cross section plane
- Sagittal rays:
perpendicular to main cross section plane
- Coma rays:
Going through field point
and edge of pupil
- Oblique rays:
without symmetry



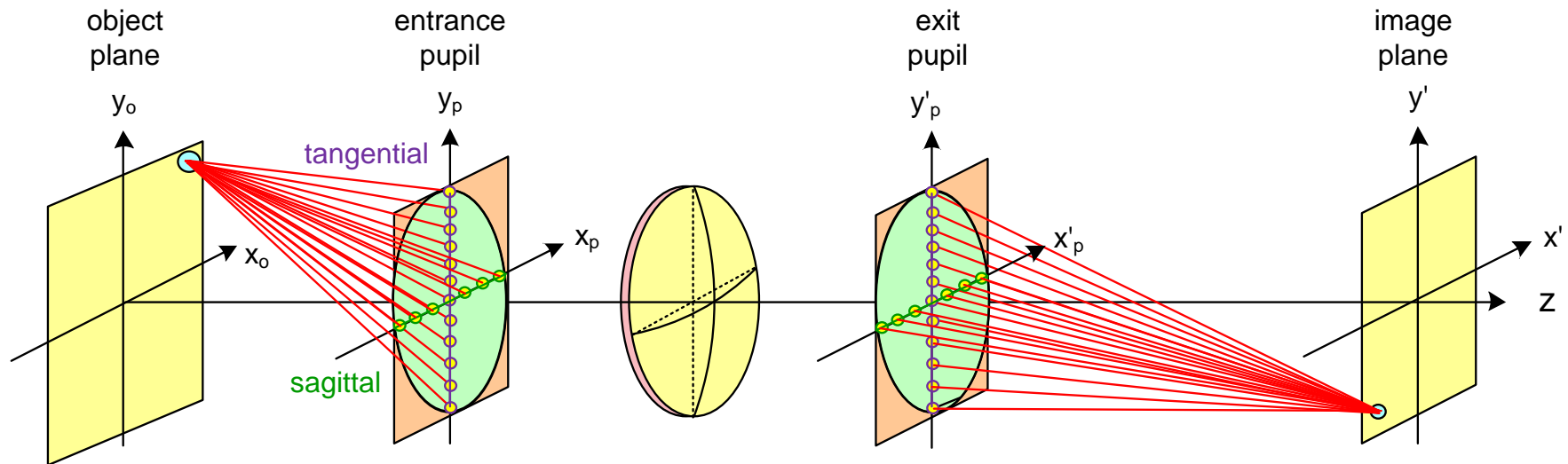


Ray-Fan Selection for Transverse Aberration Plots

- Transverse aberrations:
Ray deviation from ideal image point in meridional and sagittal plane respectively
- The sampling of the pupil is only filled in two perpendicular directions along the axes
- No information on the performance of rays in the quadrants of the pupil

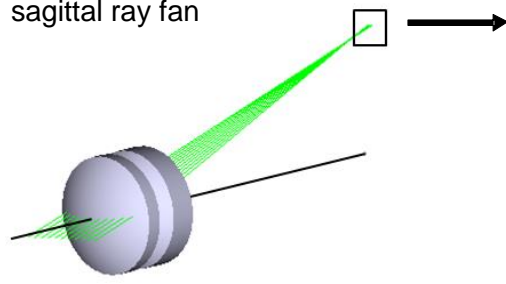


- Pupil sampling for calculation of transverse aberrations:
all rays from one object point to all pupil points on x- and y-axis
- Two planes with 1-dimensional ray fans
- No complete information: no skew rays

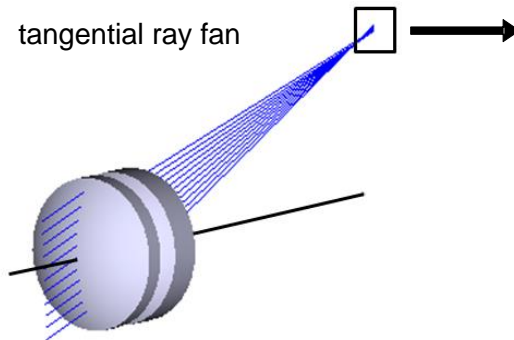


- Ray plots
- Spot diagrams

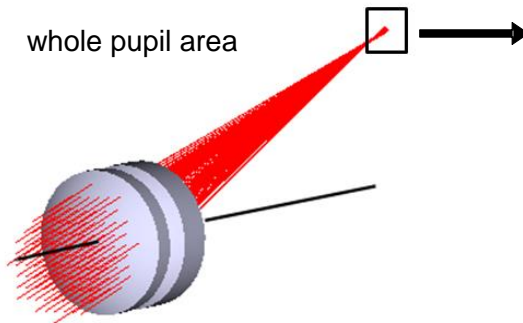
sagittal ray fan



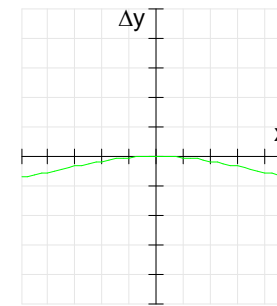
tangential ray fan



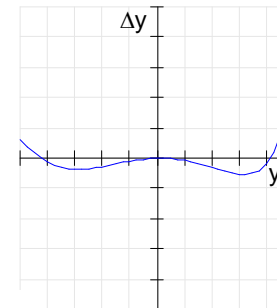
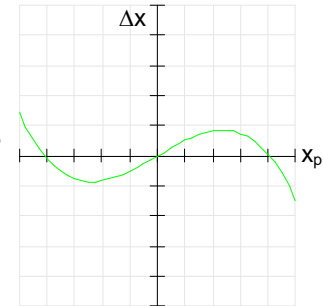
whole pupil area



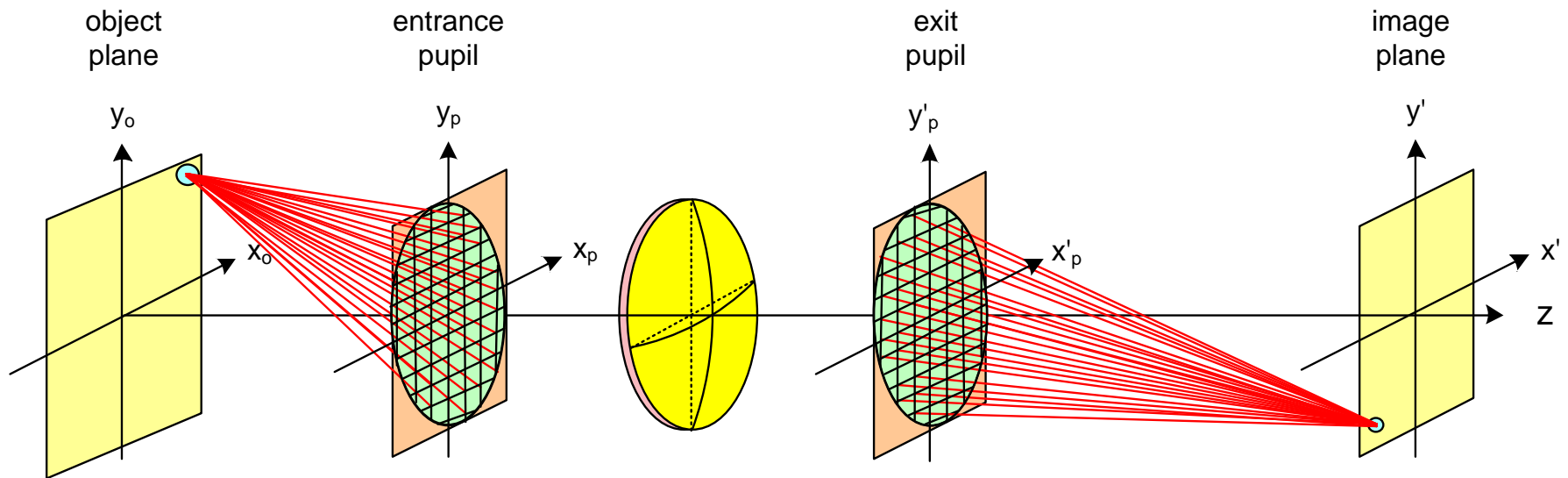
tangential aberration



sagittal aberration

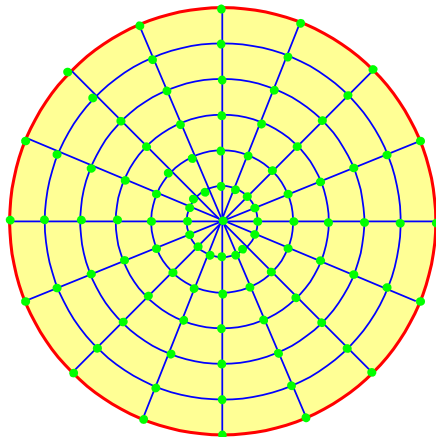


- Pupil sampling in 3D for spot diagram:
all rays from one object point through all pupil points in 2D
- Light cone completely filled with rays

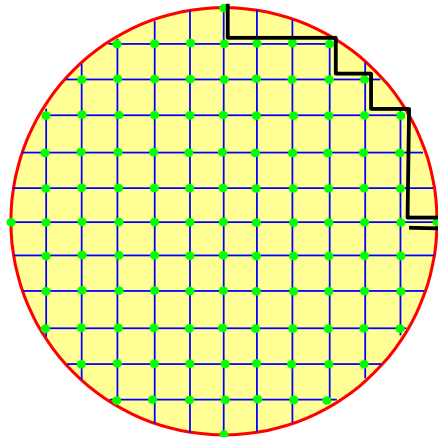


- Criteria:
 1. iso energetic rays
 2. good boundary description
 3. good spatial resolution

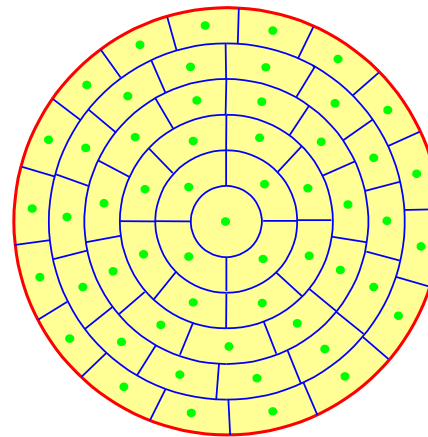
polar grid



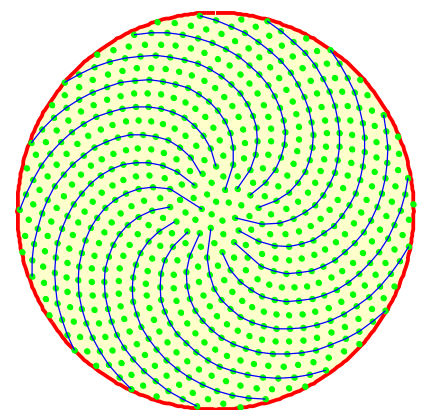
cartesian



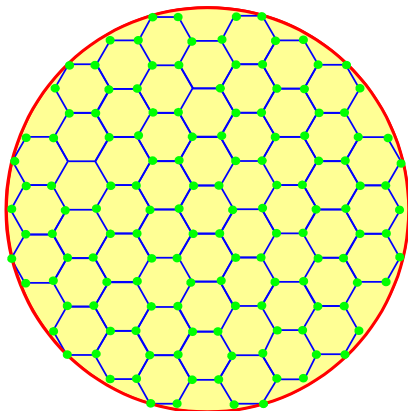
isoenergetic circular



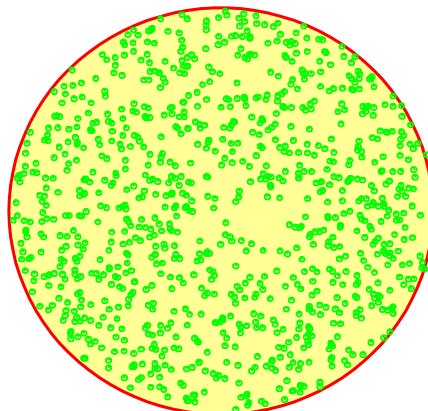
Fibonacci spirals



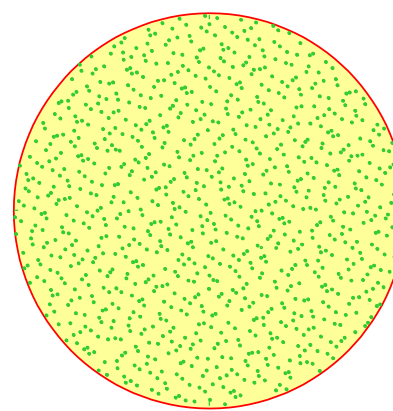
hexagonal



statistical

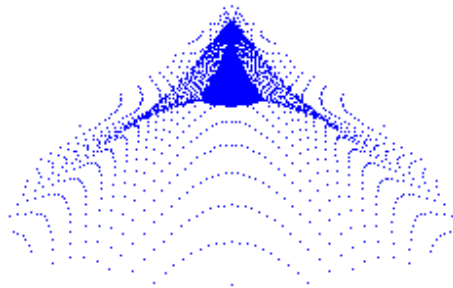


pseudo-statistical (Halton)

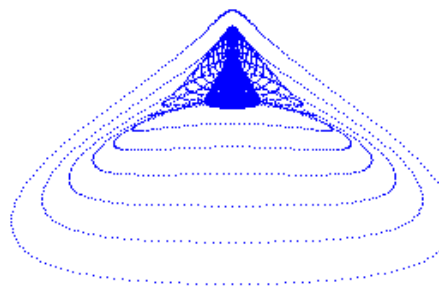


- Artefacts due to regular gridding of the pupil of the spot in the image plane
- In reality a smooth density of the spot is true
- The line structures are discretization effects of the sampling

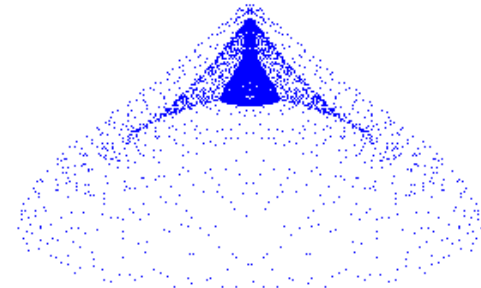
cartesian



hexagonal

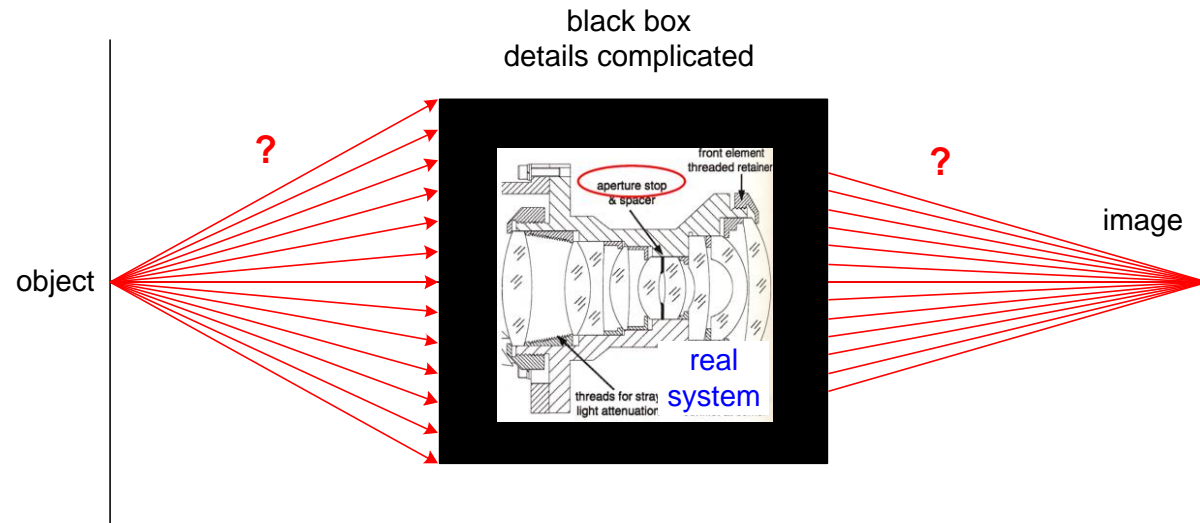


statistical

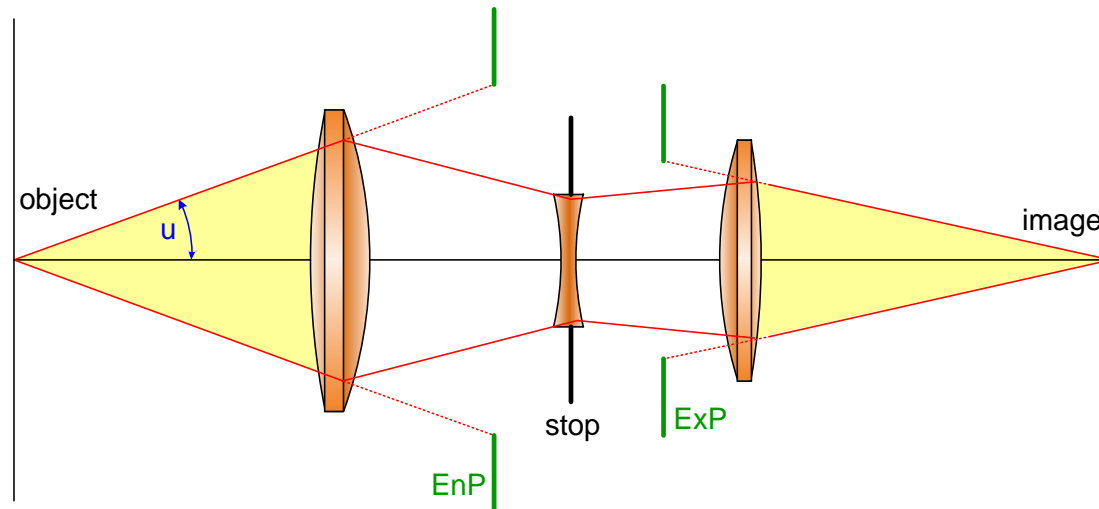


Diaphragm in Optical Systems

- The physical stop defines the aperture cone angle u
- The real system may be complex



- The entrance pupil fixes the acceptance cone in the object space
- The exit pupil fixes the acceptance cone in the image space





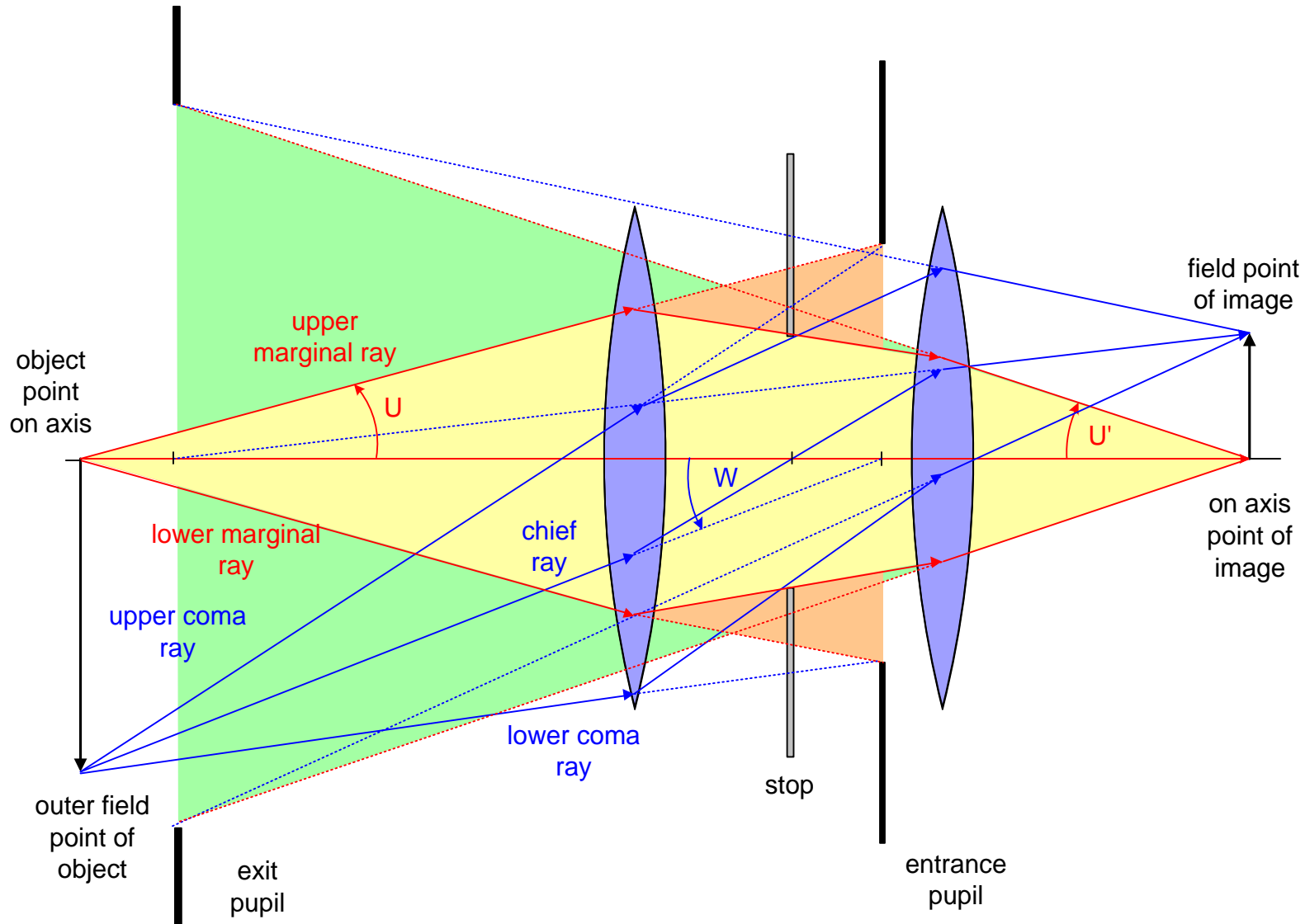
Properties of the Pupil

Relevance of the system pupil :

- Brightness of the image
Transfer of energy
- Resolution of details
Information transfer
- Image quality
Aberrations due to aperture
- Image perspective
Perception of depth
- Compound systems:
matching of pupils is necessary, location and size



Entrance and Exit Pupil



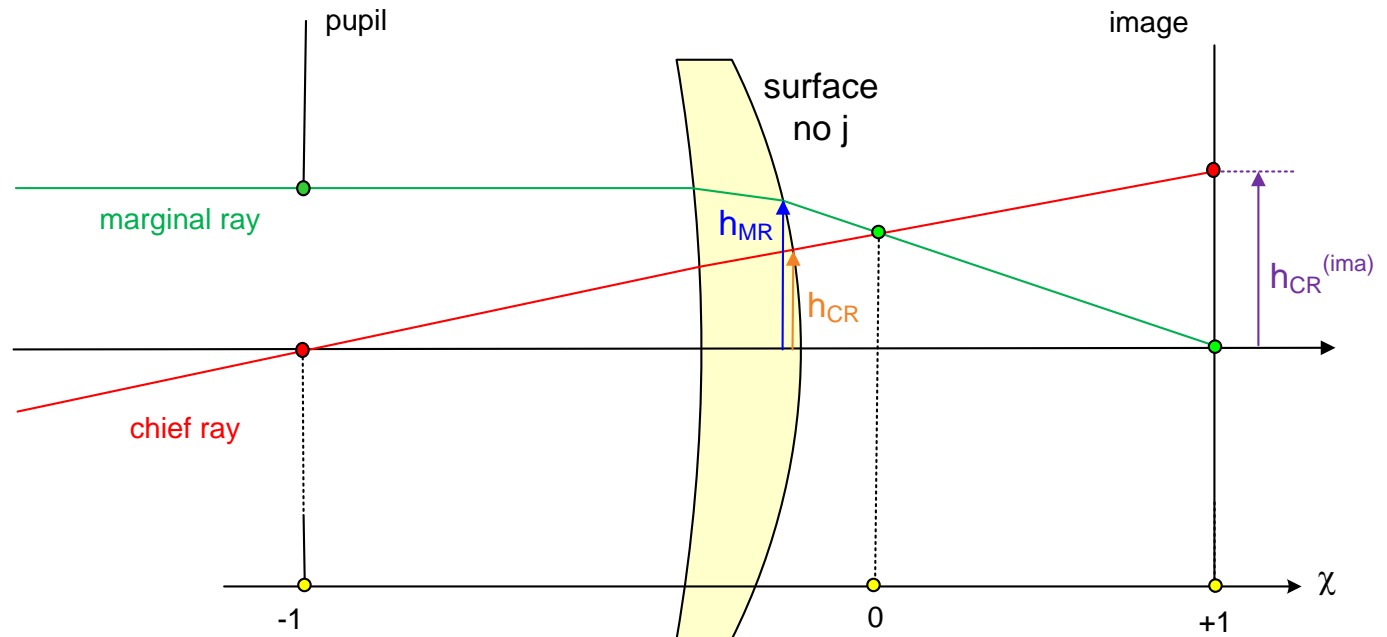


Parameter of Excentricity

- Relative position of stop inside system
- Quantitative measure:
Parameter of excentricity
(h absolut values)

$$\chi = \frac{h_{CR} - h_{MR}}{h_{CR} + h_{MR}}$$
- Special cases:

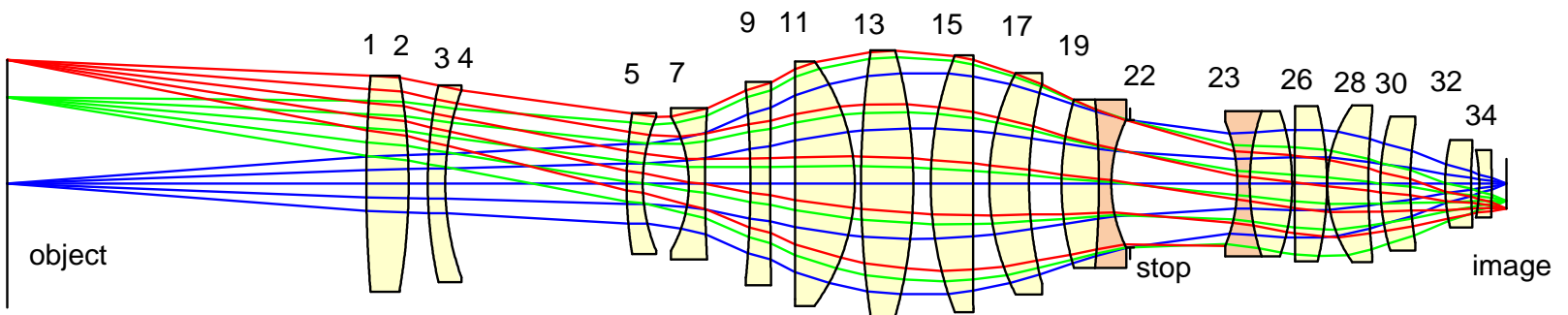
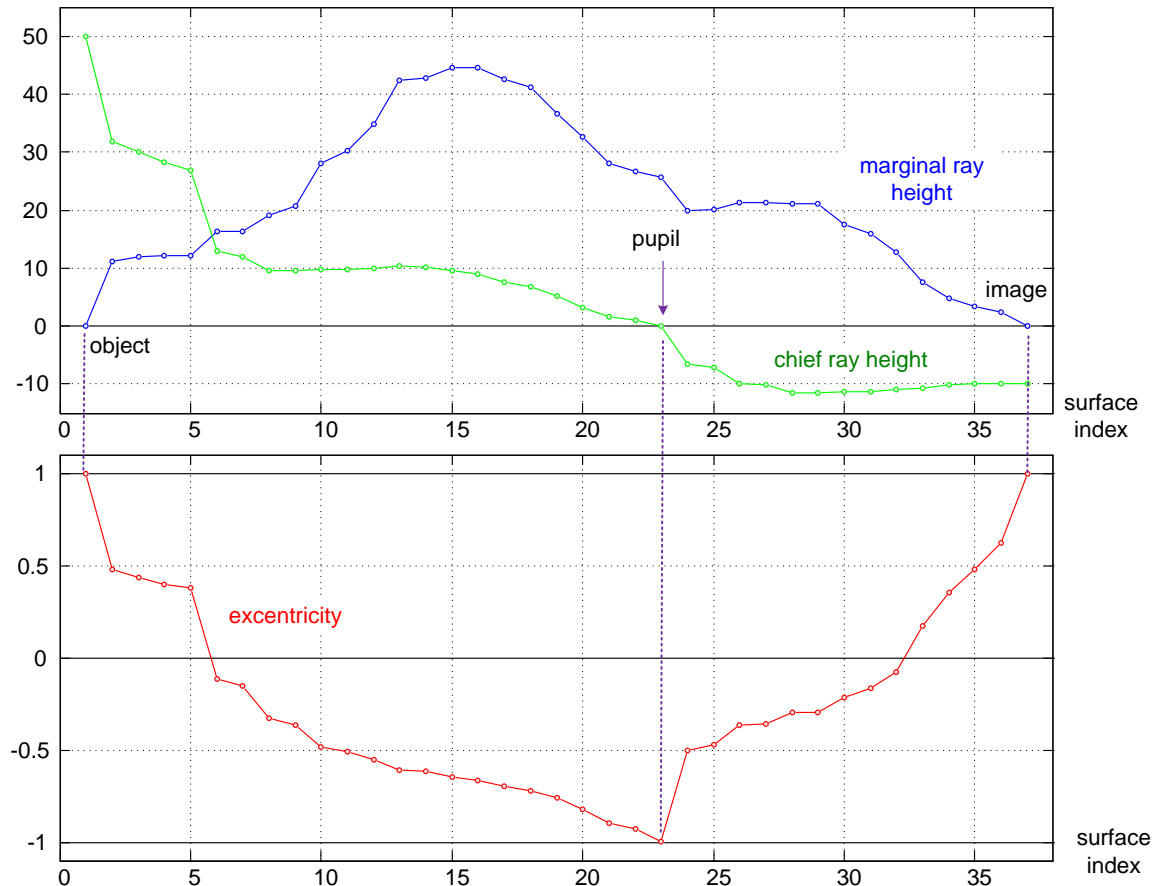
$\chi = 1$	image plane
$\chi = -1$	pupil plane
$\chi = 0$	same effective distance from image and pupil



Parameter of Excentricity



- Example:
excentricity for all surfaces
- Change:
 $\chi = +1 \dots -1 \dots +1$



- Telescopic observation with different f-numbers
- Bad match of pupil location: key hole effect

$F\# = 2.8$

$F\# = 8$

$F\# = 22$

a) pupil
adapted

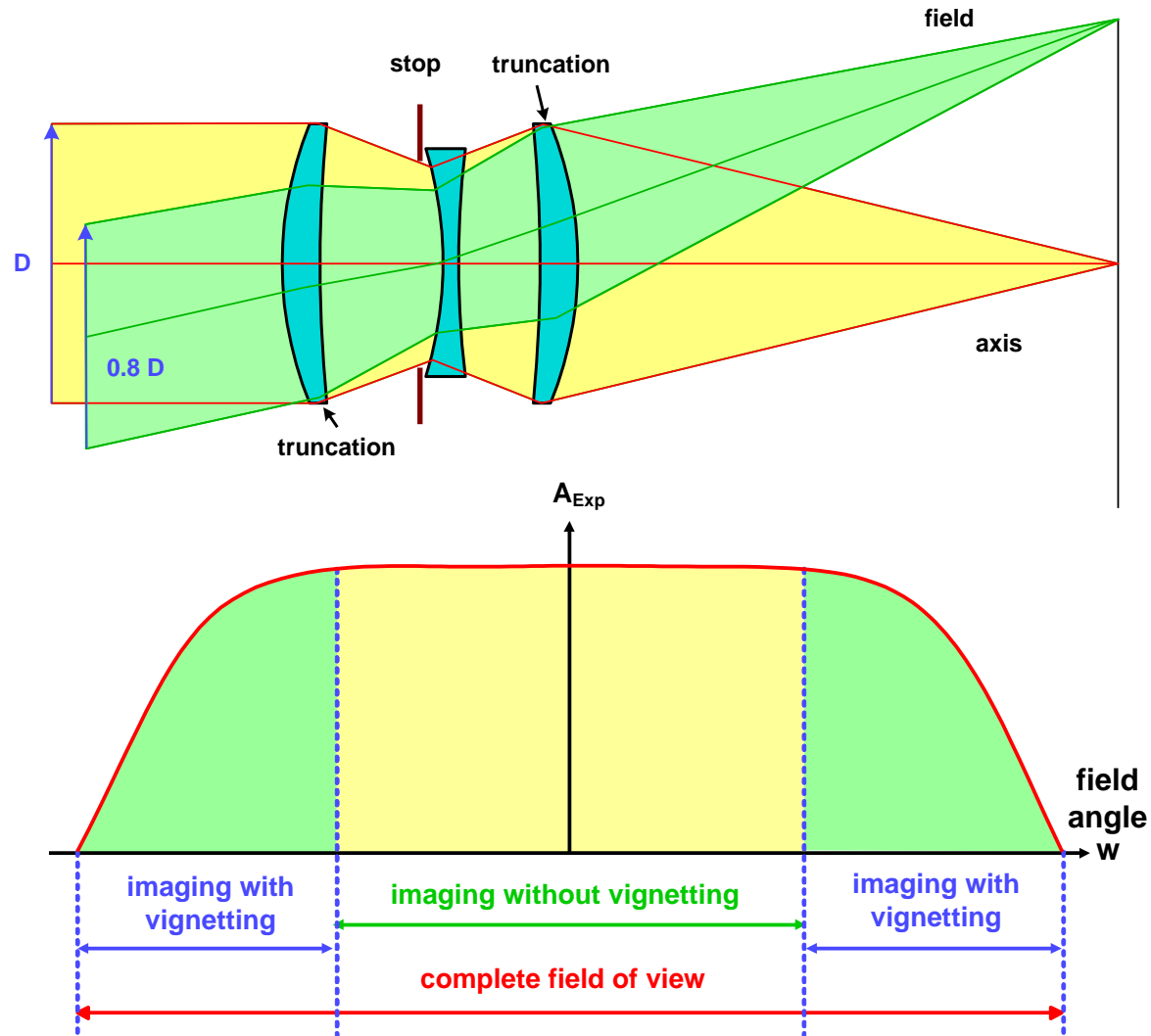


b) pupil
location
mismatch

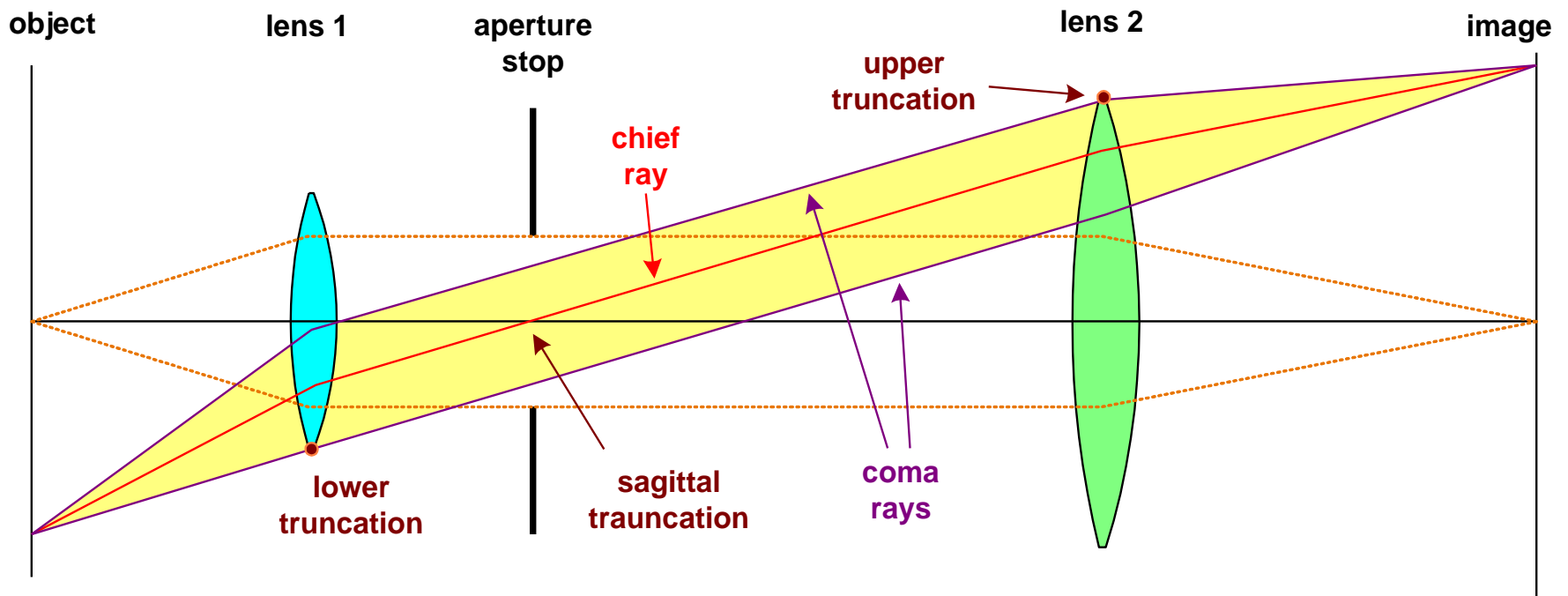


Ref: H. Schlemmer

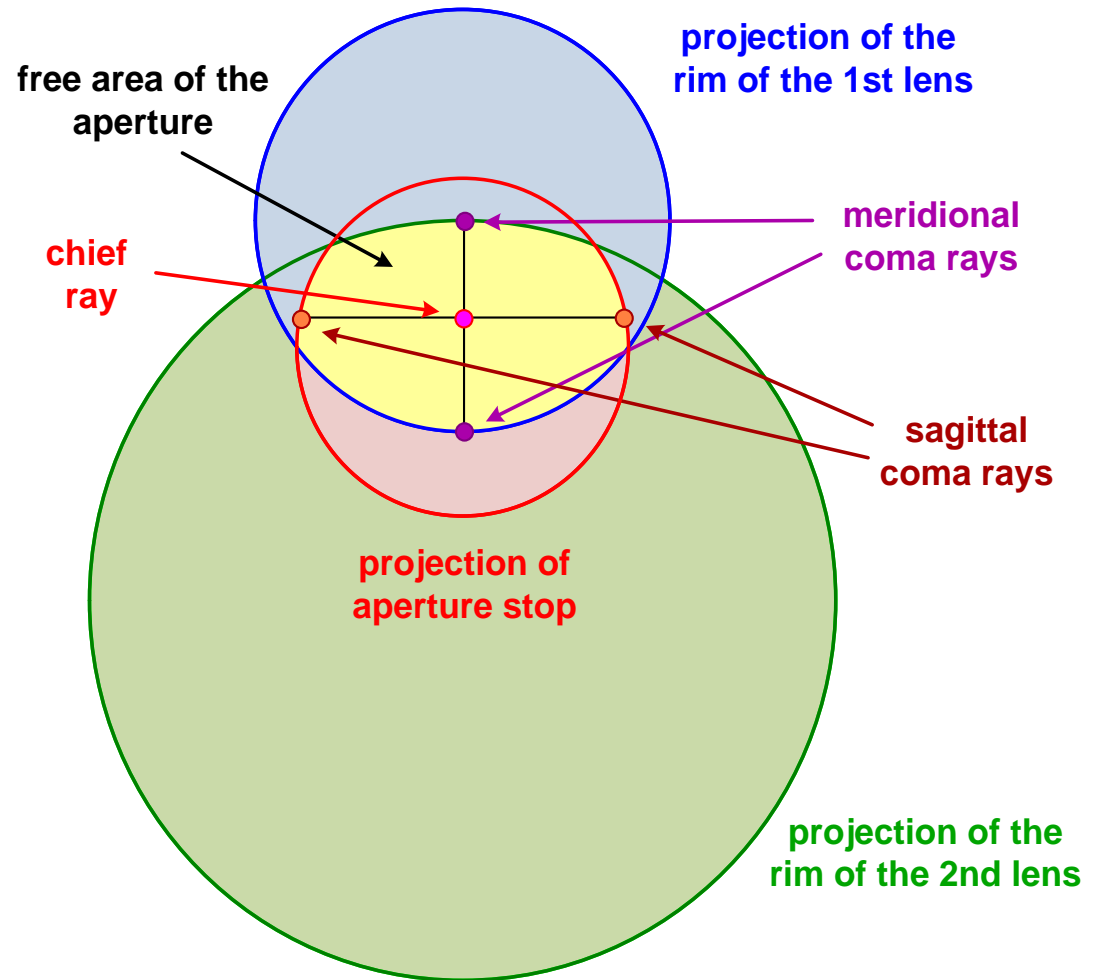
- Artificial vignetting:
Truncation of the free area of the aperture light cone
- Natural Vignetting:
Decrease of brightness according to $\cos^4 w$ due to oblique projection of areas and changed photometric distances



- 3D-effects due to vignetting
- Truncation of the cone at different surfaces for the upper and the lower part of the cone



- Truncation of the light cone with asymmetric ray path for off-axis field points
- Intensity decrease towards the edge of the image
- Definition of the chief ray: ray through energetic centroid
- Vignetting can be used to avoid uncorrectable coma aberrations in the outer field
- Effective free area with extrem aspect ratio: anamorphic resolution



- Illumination fall off in the image due to vignetting at the field boundary





Helmholtz-Lagrange Invariant

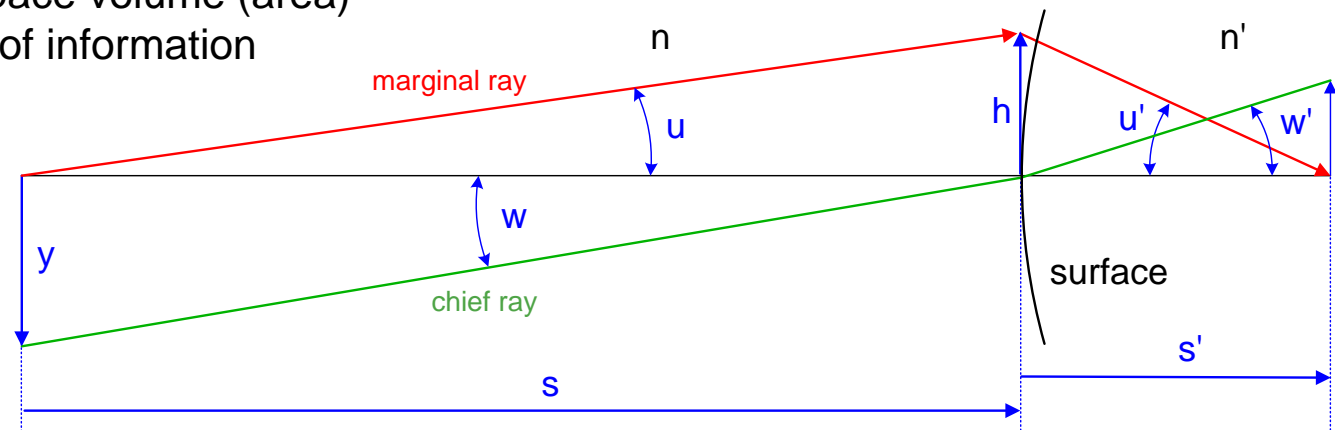
- Product of field size y and numerical aperture is invariant in a paraxial system
- Derivation at a single refracting surface:
 - Common height h :
 - Triangles
 - Refraction:
 - Elimination of s, s', w, w'
- The invariance corresponds to:
 - Energy conservation
 - Liouville theorem
 - Invariant phase space volume (area)
 - Constant transfer of information

$$L = n \cdot y \cdot u = n' \cdot y' \cdot u'$$

$$h = s \cdot u = s' \cdot u'$$

$$w = \frac{y}{s}, \quad w' = \frac{y'}{s'}$$

$$nw = n'w'$$



Helmholtz-Lagrange Invariant

- Basic formulation of the Lagrange invariant:
Uses image height,
only valid in field planes

- General expression:

- Triangle SPB

$$w' = \frac{y_{CR}}{s'_{Exp}}$$

- Triangle ABO'

$$y'_{CR} = w' \cdot (s' - s'_{Exp})$$

- Triangle SQA

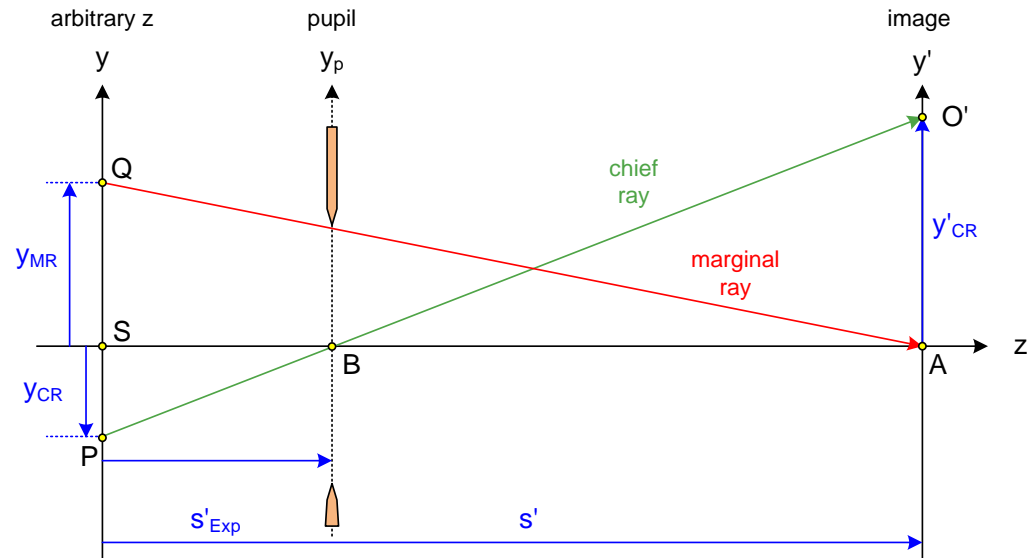
$$u' = \frac{y_{MR}}{s'}$$

- Gives

$$L = n' \cdot u' \cdot y'_{CR} = n' \cdot \frac{y_{MR}}{s'} \cdot w' \cdot (s' - s'_{Exp}) = n' \cdot (y_{MR} w' - u' w' s'_{Exp})$$

- Final result for arbitrary z:

$$L = n' \cdot [w' \cdot y_{MR}(z) - u' \cdot y_{CR}(z)]$$

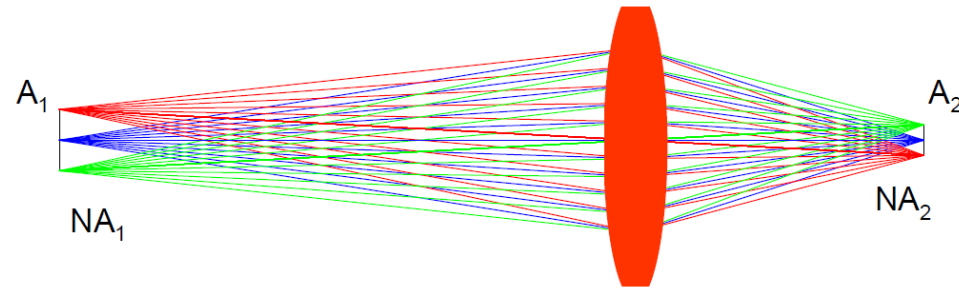


- Simple definition as Lagrange invariant:

$$L_{parax} = A \cdot NA^2$$

Assumptions:

- paraxial approximation
- circular symmetric cross section
- uniform filling of the pupil



- Geometrical generalized definition:

$$L_{geo} = \iiint dx dy dp dq$$

- projected solid angle
- pupil can have an arbitrary boundary, but is uniform illuminated

$$p = n \cdot \cos(u) \quad , \quad q = n \cdot \cos(v)$$



- Physical definition:

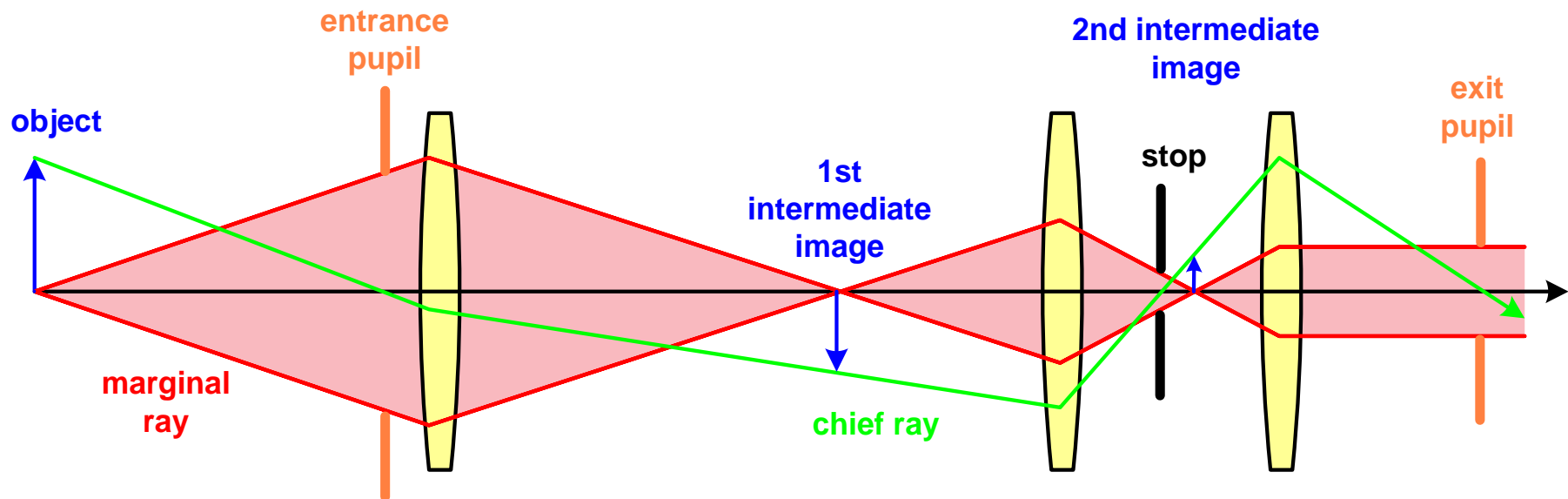
$$L_{phys} = \iiint \int E d\lambda dx dy dp dq$$

- radiance E as function of wavelength, polarization, spatial and angular coordinates
- generalized representation as spectral density in phase space, Wigner function

Nested Ray Path

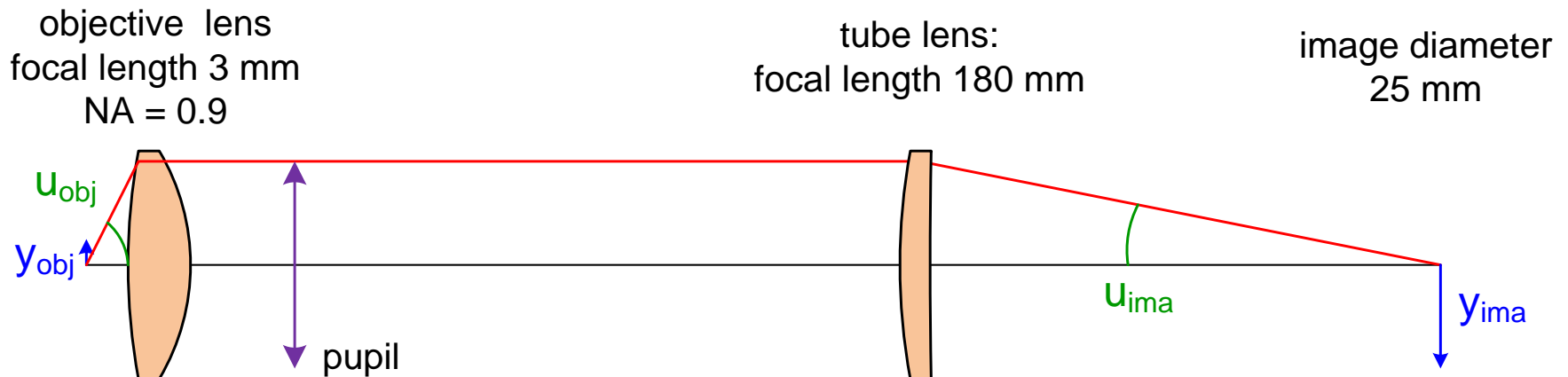
Optical Image formation:

- Sequence of pupil and image planes
- Matching of location and size of image planes necessary (trivial)
- Matching of location and size of pupils necessary for invariance of energy density
- In microscopy known as Köhler illumination



Helmholtz-Lagrange Invariant

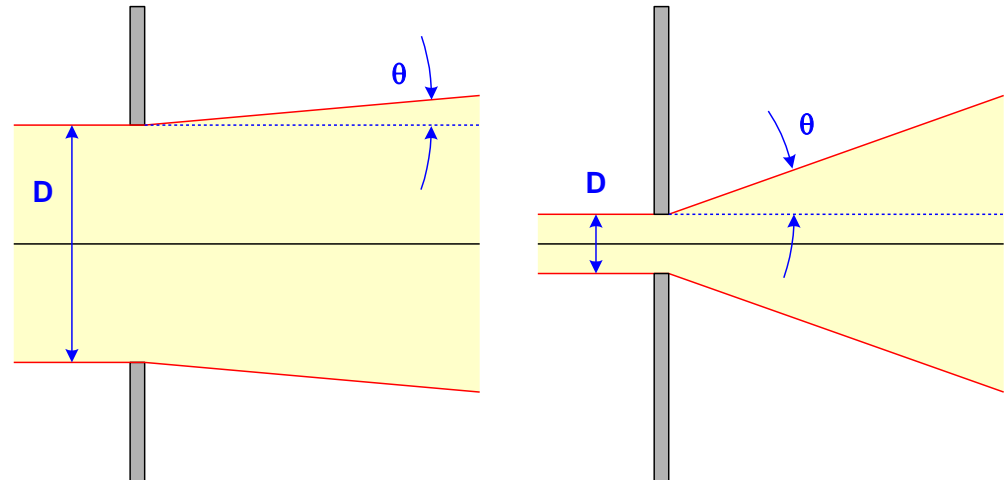
- Simple example:
 - A microscope is a 4f-system with objective lens ($f_{obj} = 3 \text{ mm}$) and tube lens ($f_{TL} = 180 \text{ mm}$)
 - the numerical aperture is $NA = 0.9$ and the intermediate Image size $D = 2y_{ima} = 25 \text{ mm}$
 - magnification $m = f_{TL} / f_{obj} = 60$
 - image sided aperture $u_{ima} = u_{obj} / m = 0.015$
 - pupil size $D_{pup} = f_{obj} \cdot NA = 2.7 \text{ mm}$
 - object field $y_{obj} = y_{ima} \cdot u_{ima} / u_{obj} = 0.42 \text{ mm}$



1. Slit diffraction

Diffraction angle inverse to slit width D

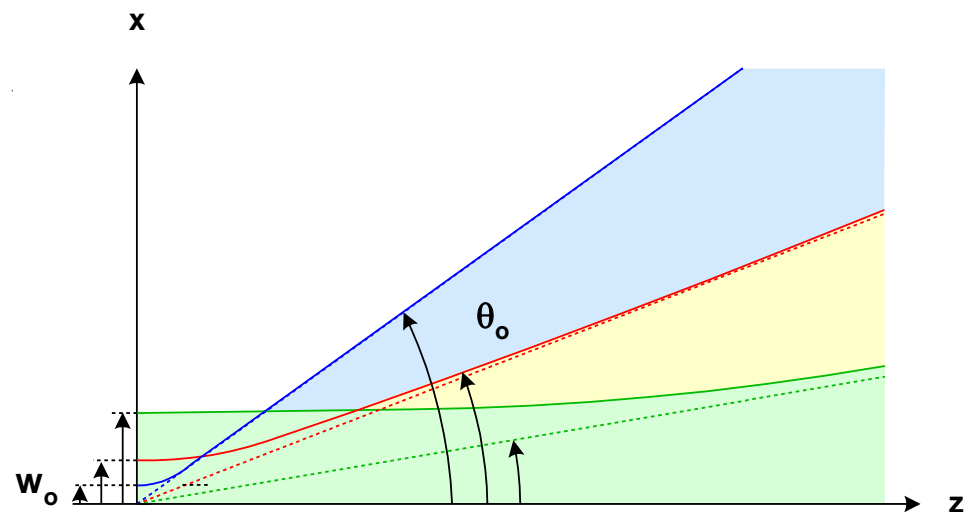
$$\theta = \frac{\lambda}{D}$$



2. Gaussian beam

Constant product of waist size w_0 and divergence angle θ_0

$$w_0 \theta_0 = \frac{\lambda}{\pi}$$





Helmholtz-Lagrange Invariant

- Laser optics: beam parameter product
waist radius times far field divergence angle
- Minimum value of L:
TEM₀₀ - fundamental mode
- Elementary area of phase space:
Uncertainty relation in optics
- Laser modes: discrete structure of phase space
- Geometrical optics: quasi continuum
- L is a measure of quality of a beam
small L corresponds to a good focussability

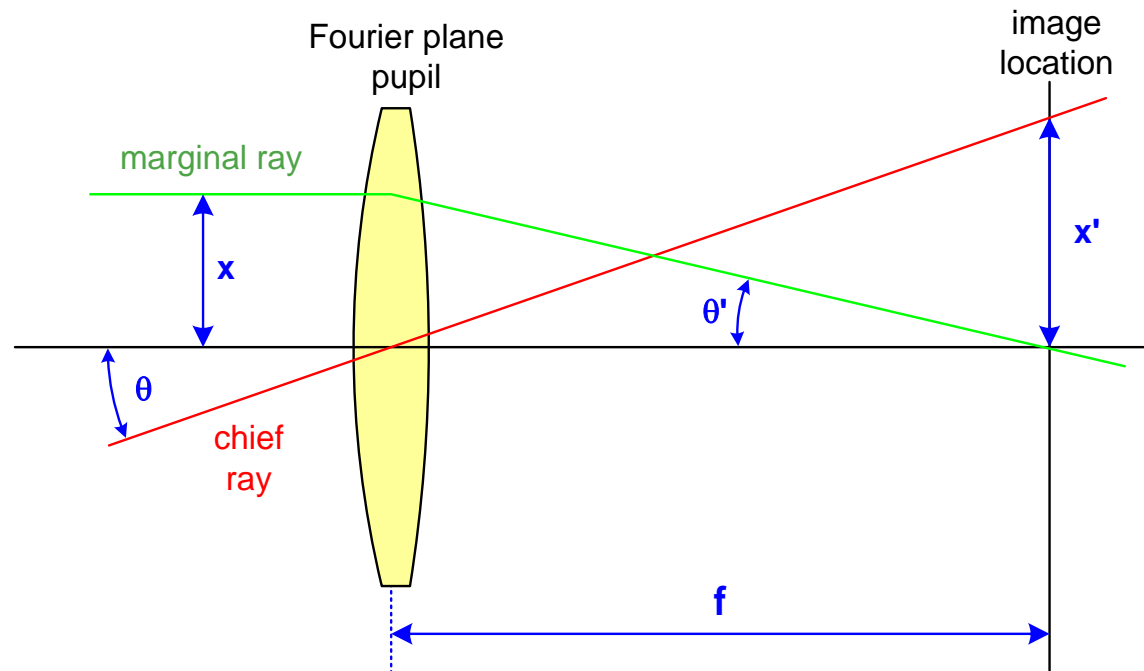
$$L_{GB} = w_o \cdot \theta_o$$

$$L_{GB} = \frac{\lambda}{\pi}$$

$$L_{GB} = w_n \cdot \theta_n = \frac{\lambda}{\pi} \cdot (2n+1)$$

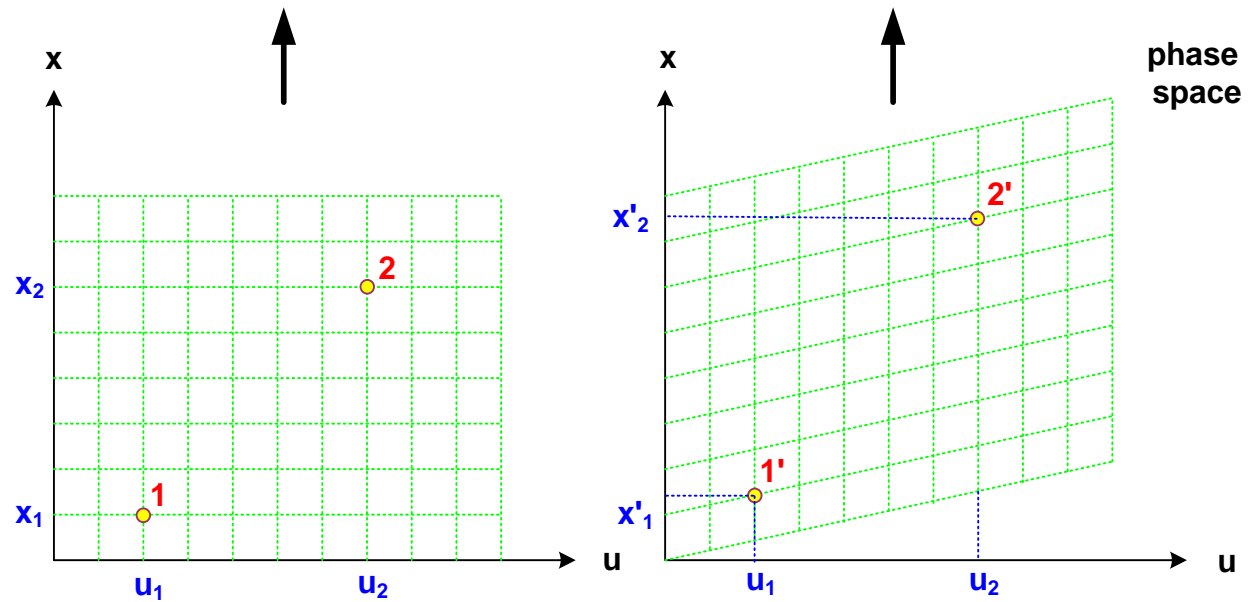
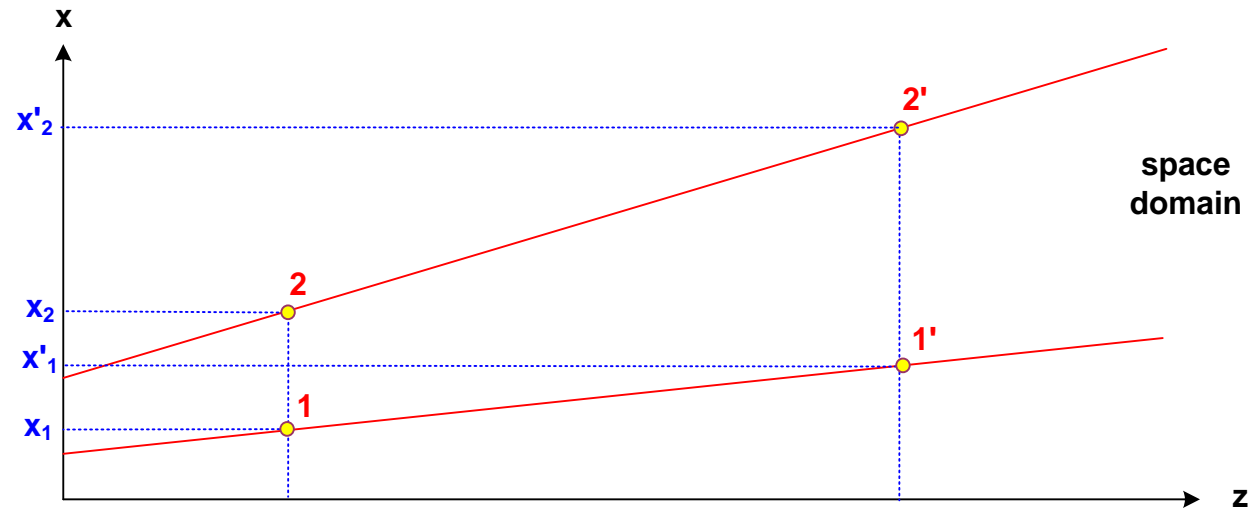
Phase Space: 90° -Rotation

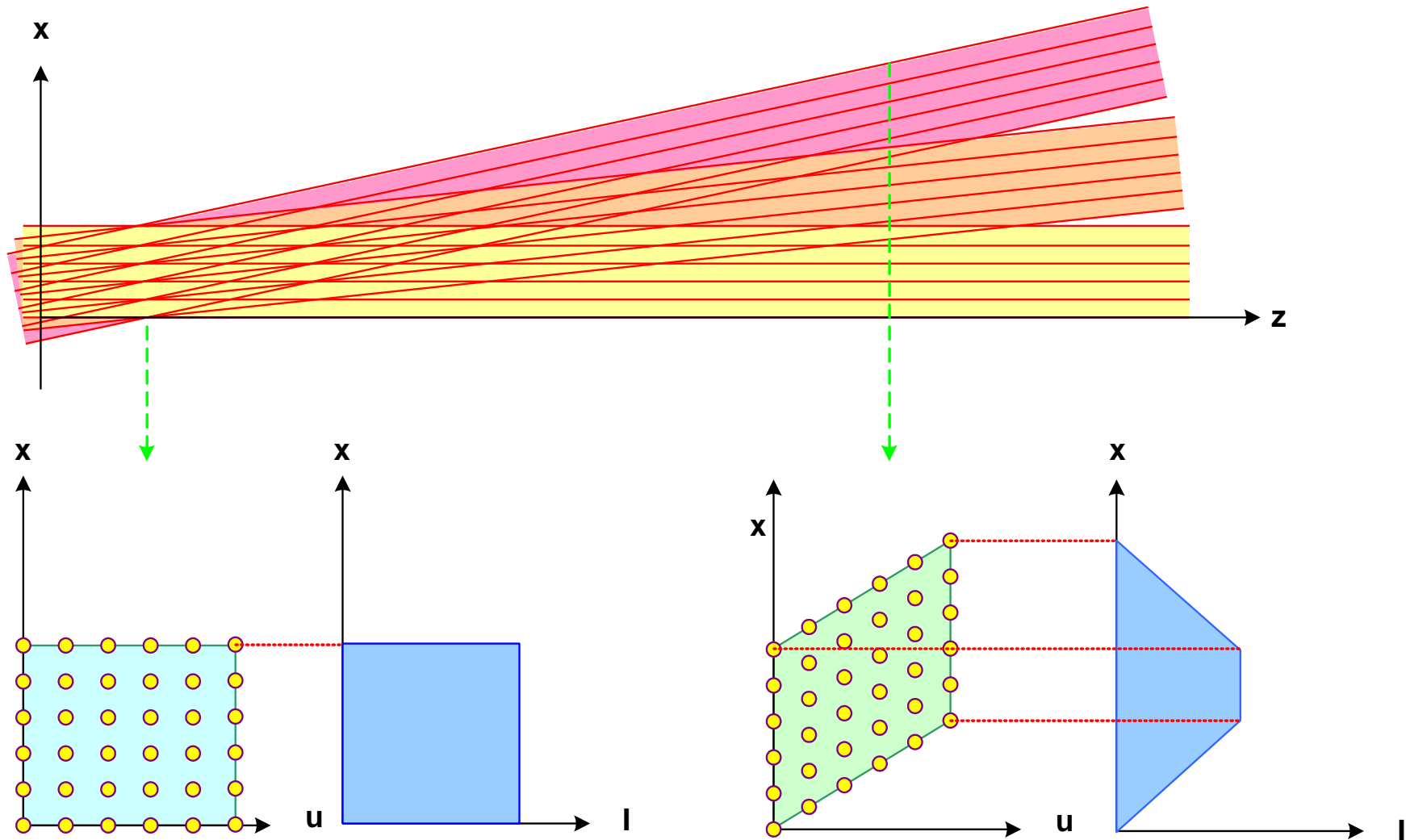
- Transition pupil-image plane: 90° rotation in phase space
- Planes Fourier inverse
- Marginal ray: space coordinate $x \rightarrow$ angle θ'
- Chief ray: angle $\theta \rightarrow$ space coordinate x'



Phase Space

Direct phase space
representation of raytrace:
spatial coordinate vs angle

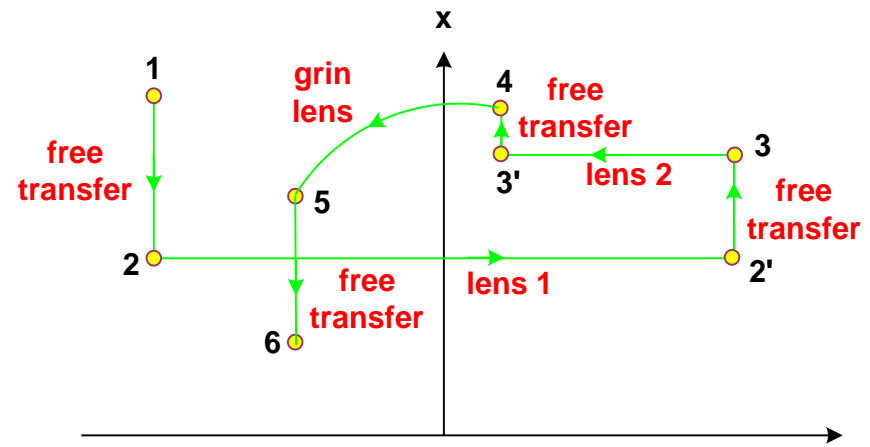
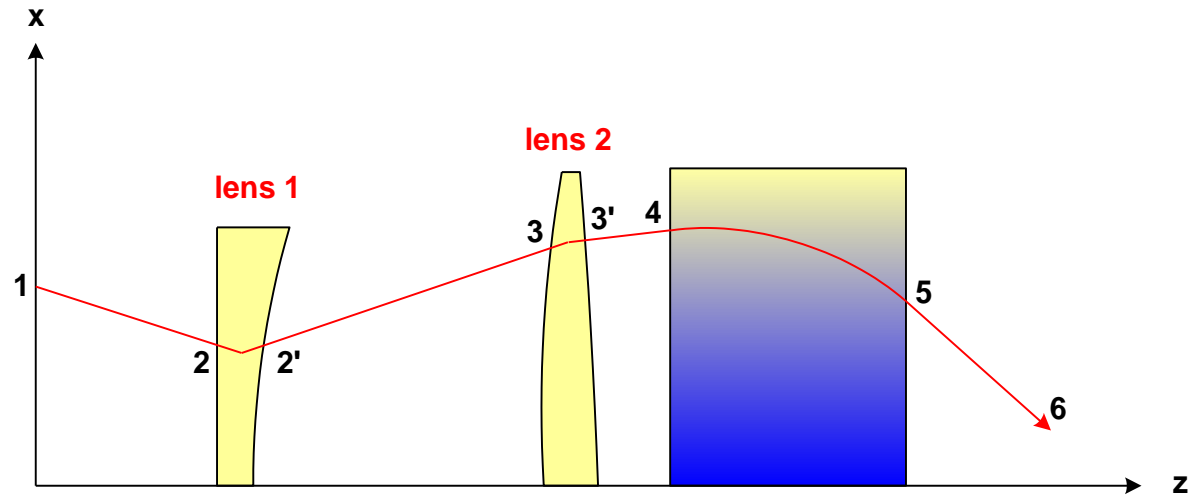




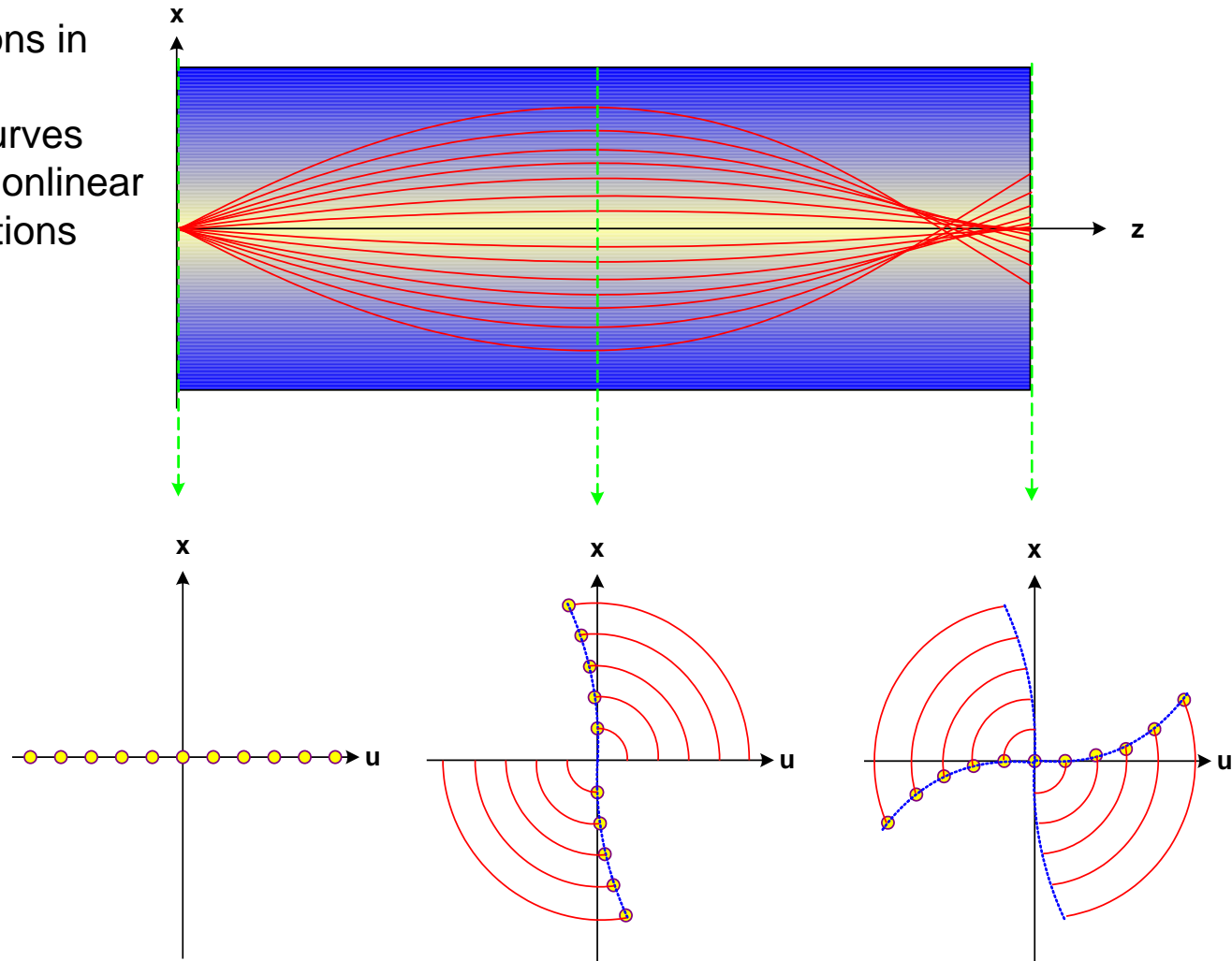


Phase Space

Direct phase space
representation of raytrace:
spatial coordinate vs angle



- Grin lens with aberrations in phase space:
 - continuous bended curves
 - aberrations seen as nonlinear angle or spatial deviations





Transfer of Energy in Optical Systems

- Conservation of energy

$$d^2\Phi = d^2\Phi'$$

- Differential flux (L: radiance)

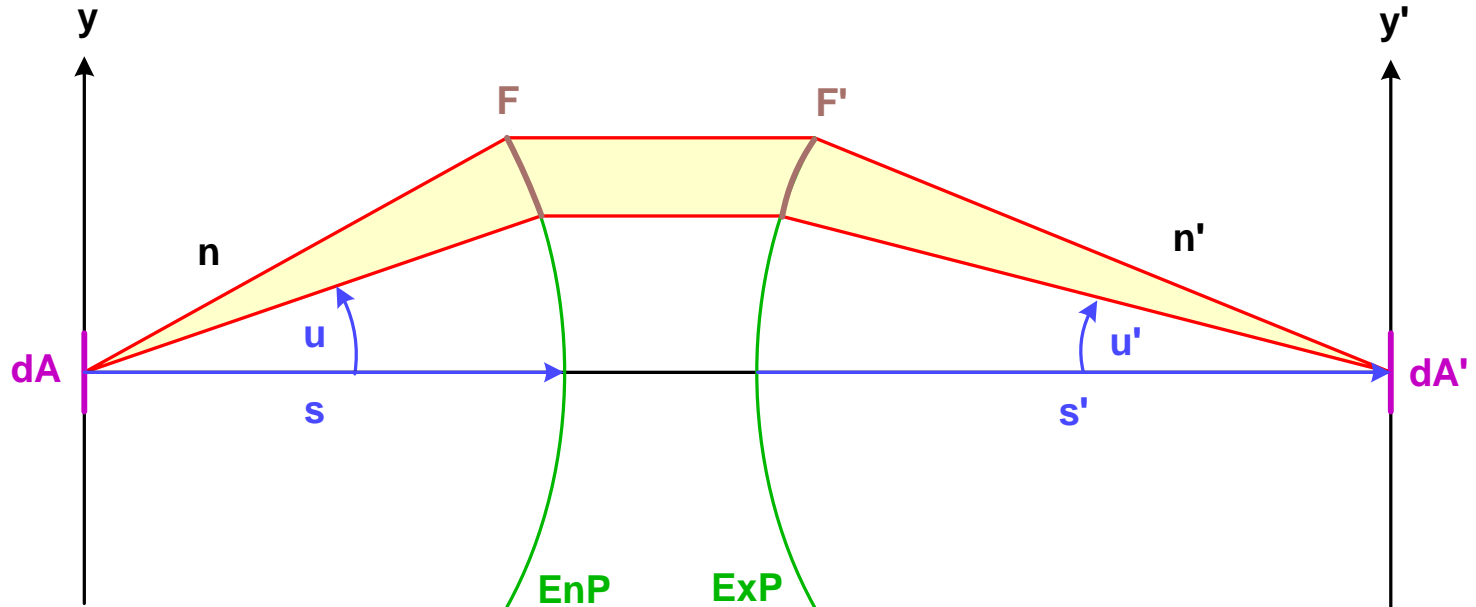
$$d^2\Phi = L \cdot \sin u \cdot \cos u \cdot dA \cdot du \, d\varphi$$

- No absorption

$$T = 1$$

- Sine condition

$$n y \cdot \sin u = n' y' \cdot \sin u'$$

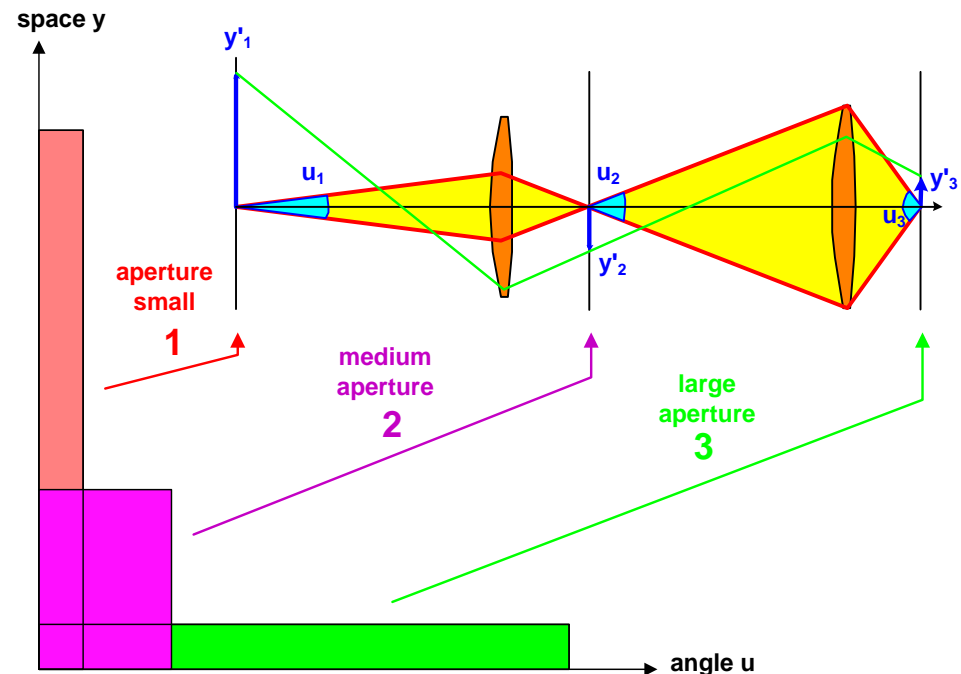


Helmholtz-Lagrange Invariant

- Geometrical optic:
Etendue, light gathering capacity
- Paraxial optic: invariant of Lagrange / Helmholtz
- General case: 2D
- Invariance corresponds to conservation of energy
- Interpretation in phase space:
constant area, only shape is changed at the transfer through an optical system

$$L_{Geo} = \frac{D_{field}}{2} \cdot n \cdot \sin u$$

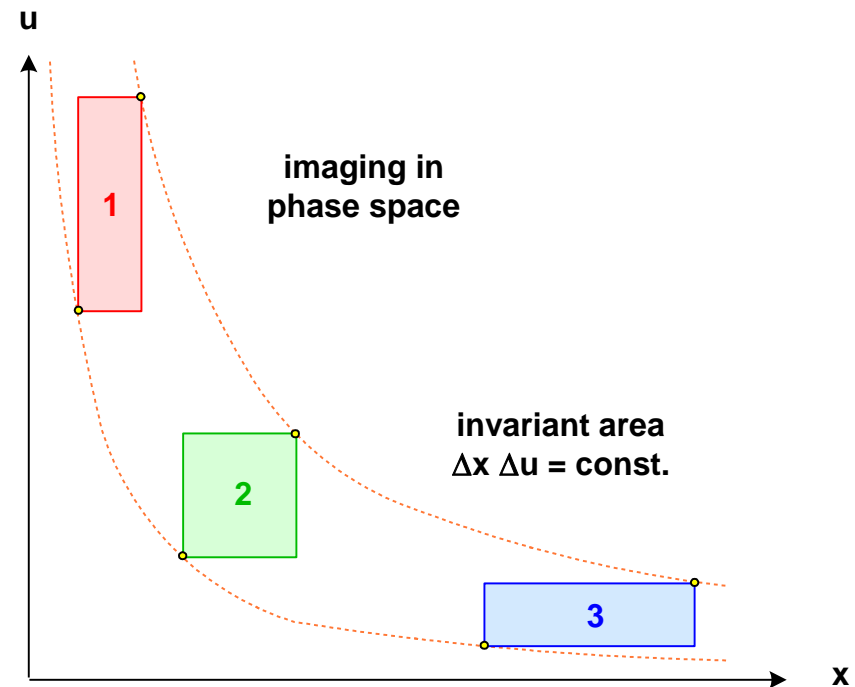
$$L = n \cdot y \cdot u = n' \cdot y' \cdot u'$$



- Invariance of Energy:
 - constant area in phase space in the geometrical model
 - constant integral over density in the wave optical model

- Incoherent ensemble, quasi continuum:
Jacobian matrix of transformation relates
the coordinate changes

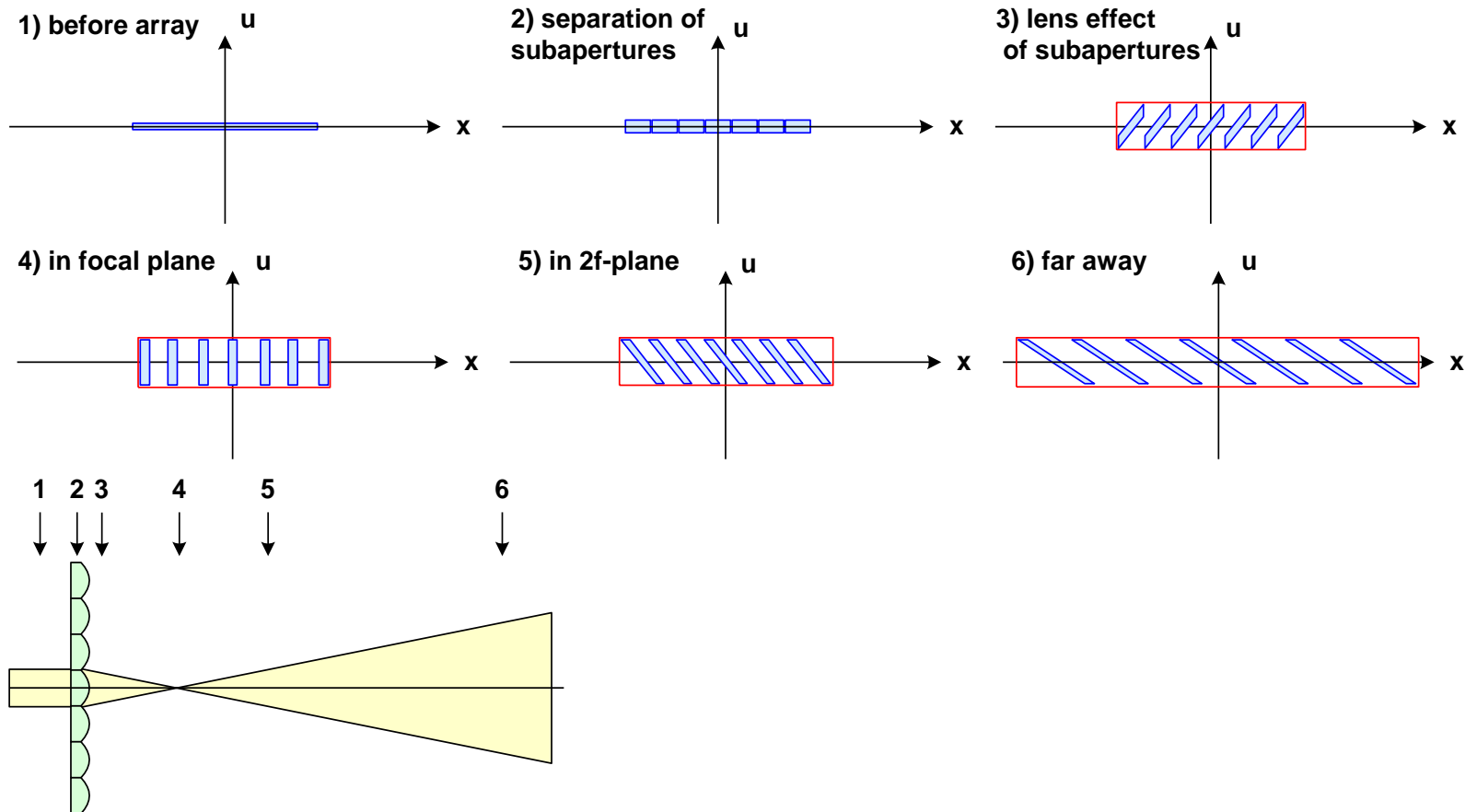
$$J = \frac{\partial L}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial L}{\partial y} \cdot \frac{\partial v}{\partial x}$$



Example Phase Space of an Array

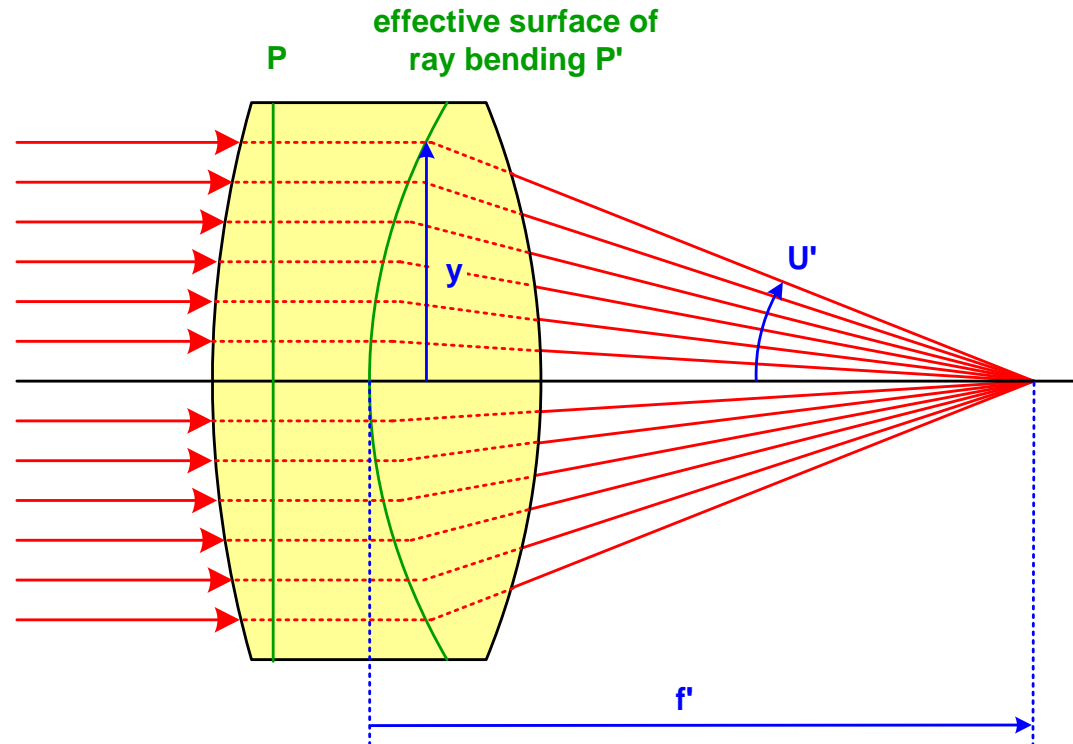
Simplified pictures to the changes of the phase space density.

Etendue is enlarged, but no complete filling.



Principal Sphere

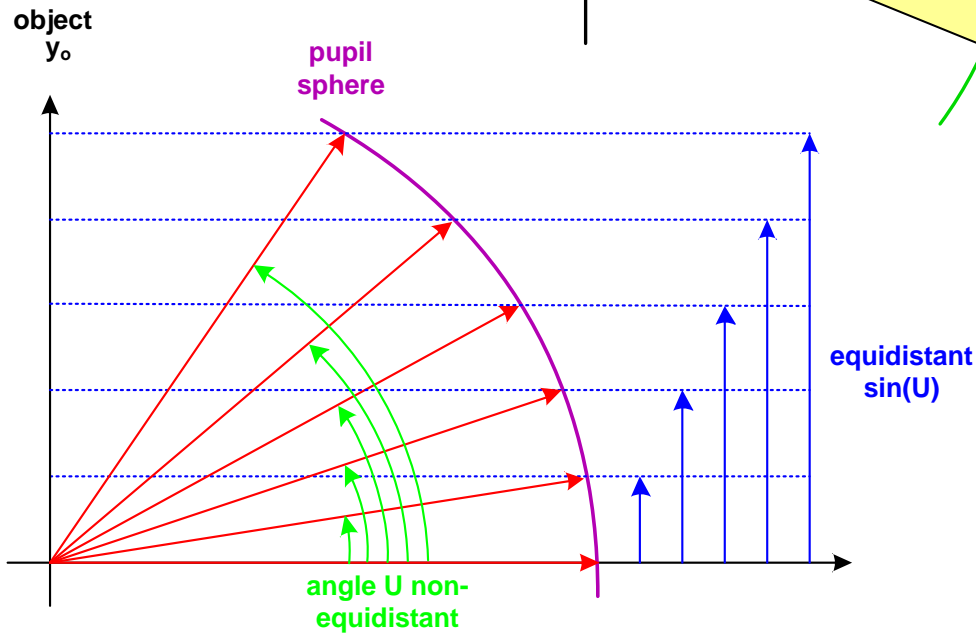
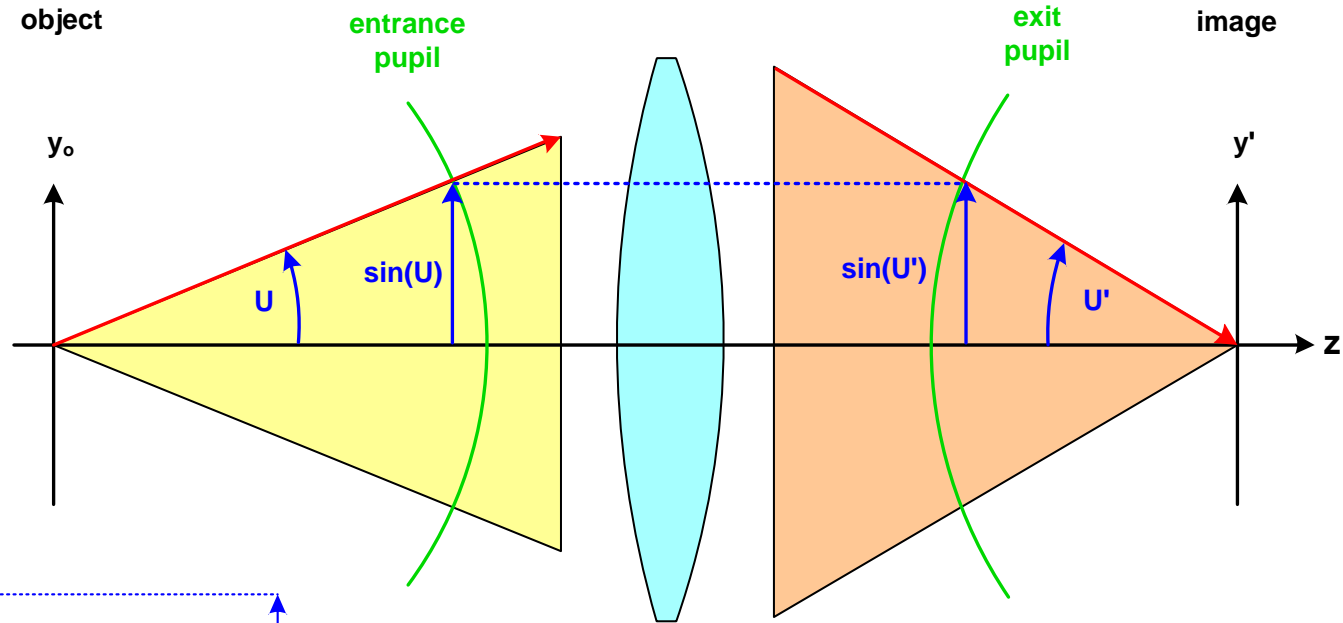
- Generalization of paraxial picture:
Principal surface works as effective location of ray bending for object points near the optical axis (isoplanatic patch)
- Paraxial approximation: plane
Can be used for all rays to find the imaged ray
- Real systems with corrected sine-condition (aplanatic):
principal sphere
- The principal sphere can not be used to construct arbitrary ray paths
- If the sine correction is not fulfilled: more complicated shape of the arteficial surface, that represents the ray bending





Pupil Sphere

- Pupil sphere:
equidistant sine-
sampling



Canonical Coordinates

Aplanatic system:

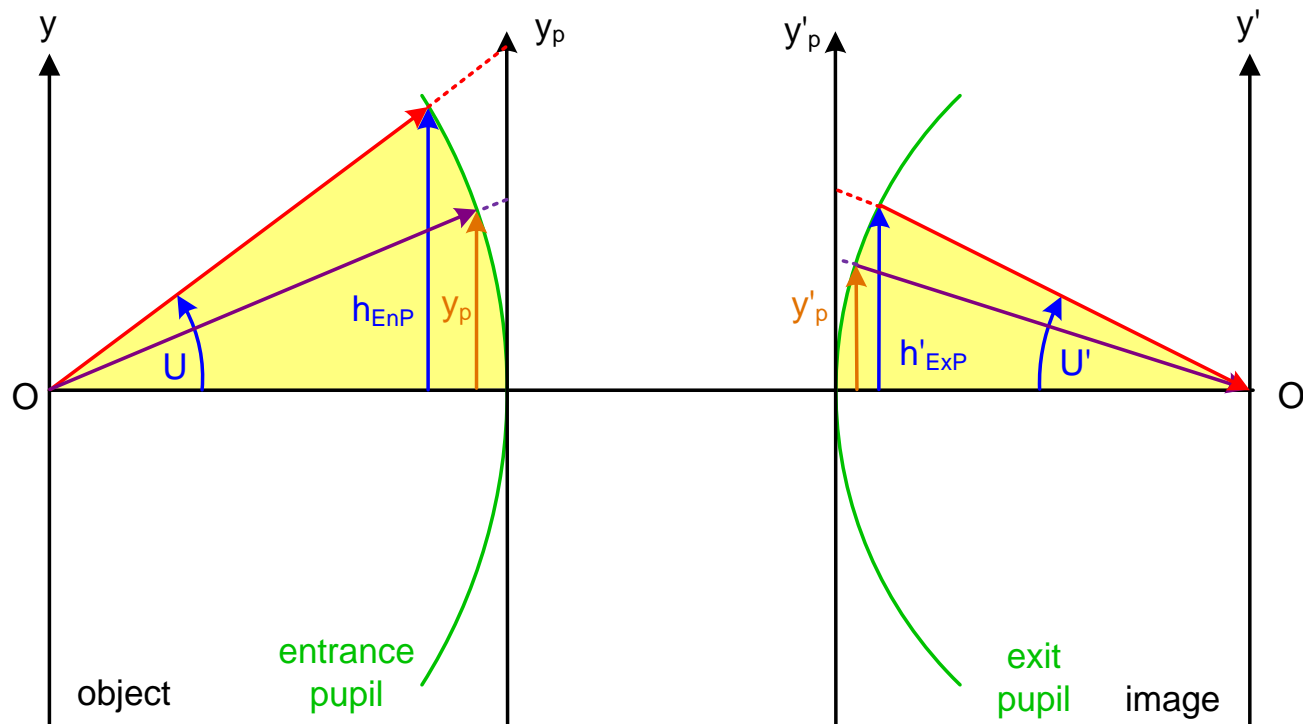
- Sine condition fulfilled
- Pupil has spherical shape
- Normalized canonical coordinates for pupil and field

$$\bar{y}_p = \frac{y_p}{h_{EnP}}$$

$$\bar{y}'_p = \frac{y'_p}{h'_{ExP}}$$

$$\bar{y} = \frac{n \cdot \sin u}{\lambda} \cdot y$$

$$\bar{y}' = \frac{n' \cdot \sin u'}{\lambda} \cdot y'$$





Canonical Coordinates

- Normalized pupil coordinates

$$\bar{x}_p = \frac{x_p}{h_{EnP}}$$

$$\bar{x}'_p = \frac{x'_p}{h'_{ExP}}$$

$$\bar{y}_p = \frac{y_p}{h_{EnP}}$$

$$\bar{y}'_p = \frac{y'_p}{h'_{ExP}}$$

- Special case: aplanatic imaging

$$\bar{x}'_p = \bar{x}_p$$

$$\bar{y}'_p = \bar{y}_p$$

- Normalized field coordinates

$$\bar{x} = \frac{n \cdot \sin u_{sag}}{\lambda} \cdot x$$

$$\bar{x}' = \frac{n' \cdot \sin u'_{sag}}{\lambda} \cdot x'$$

$$\bar{y} = \frac{n \cdot \sin u_{tan}}{\lambda} \cdot y$$

$$\bar{y}' = \frac{n' \cdot \sin u'_{tan}}{\lambda} \cdot y'$$

- Special case: paraxial imaging

$$\bar{x}' = \bar{x}$$

$$\bar{y}' = \bar{y}$$

- Reference on chief ray

$$NA = n \cdot \sin u = n \cdot (\sin u_{TCO} - \sin u_{CR})$$

- Reduced image side coordinates

$$\bar{x} = \frac{n \sin u_{sag}}{\lambda} \cdot x$$

$$\bar{y} = \frac{n \cdot (\sin u_{CR} - \sin u_{tan})}{\lambda} \cdot y$$

	Mechanics	Optics
spatial variable	x	x
Impuls variable	$p=mv$	angle u , direction cosine p , spatial frequency s_x, k_x
equation of motion spatial domain	Lagrange	Eikonal
equation of motion phase space	Hamilton	Wigner transport
potential	mass m	index of refraction n
minimal principle	Hamilton principle	Fermat principle
Liouville theorem	constant number of particles	constant energy
system function	impuls response	point spread function
wave equation	Helmholtz	Schrödinger
propagation variable	time t	space z
uncertainty	$h/2p$	$l/2p$
continuum approximation	classical mechanics	geometrical optics
exact description	quantum mechanics	wave optics