



**Institute of
Applied Physics**

Friedrich-Schiller-Universität Jena

Imaging and Aberration Theory

Lecture 3: Eikonal

2018-11-02

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Schedule - Imaging and aberration theory 2018

1	19.10.	Paraxial imaging	paraxial optics, fundamental laws of geometrical imaging, compound systems
2	26.10.	Pupils, Fourier optics, Hamiltonian coordinates	pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates
3	02.11.	Eikonal	Fermat principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media
4	09.11.	Aberration expansions	single surface, general Taylor expansion, representations, various orders, stop shift formulas
5	16.11.	Representation of aberrations	different types of representations, fields of application, limitations and pitfalls, measurement of aberrations
6	23.11.	Spherical aberration	phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders
7	30.11.	Distortion and coma	phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options
8	07.12.	Astigmatism and curvature	phenomenology, Coddington equations, Petzval law, correction options
9	14.12.	Chromatical aberrations	Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum
10	21.12.	Sine condition, aplanatism and isoplanatism	Sine condition, isoplanatism, relation to coma and shift invariance, pupil aberrations, Herschel condition, relation to Fourier optics
11	11.01.	Wave aberrations	definition, various expansion forms, propagation of wave aberrations
12	18.01.	Zernike polynomials	special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, measurement
13	25.01.	Point spread function	ideal psf, psf with aberrations, Strehl ratio
14	01.02.	Transfer function	transfer function, resolution and contrast
15	08.02.	Additional topics	Vectorial aberrations, generalized surface contributions, Aldis theorem, intrinsic and induced aberrations, reverbility

1. Fermat principle
2. Principle of stationary phase
3. Hamiltonian approach
4. Eikonal
5. Refracting surface
6. Perfect imaging
7. Ray-wave relations
8. Approximation of geometrical optics
9. Raytrace in inhomogeneous media



Fermat Principle

- Fermat principle:
the light takes the ray path, which corresponds to the shortest time of arrival
- The realized path is a minimum and therefore the first derivatives vanish

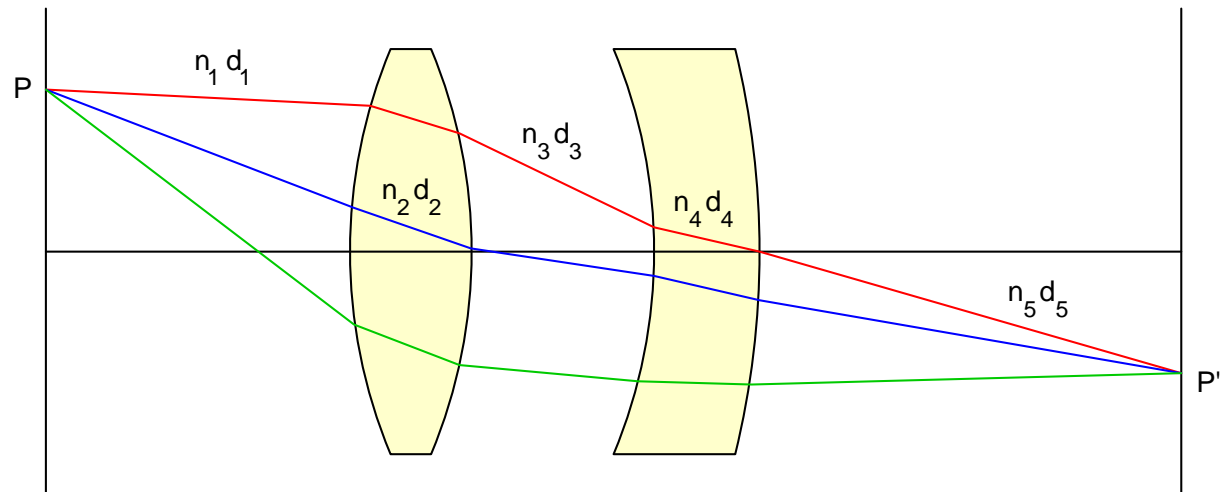
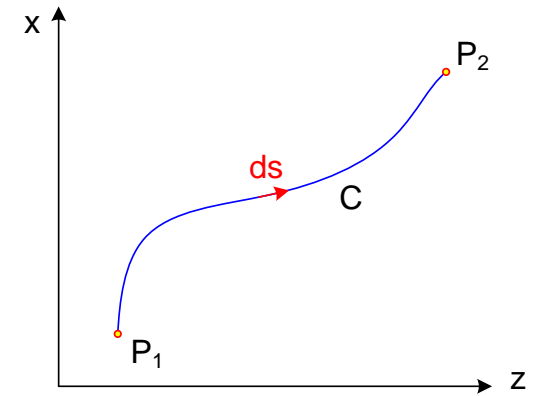
$$\delta L = \delta \int_{P_1}^{P_2} n(x, y, z) ds = 0$$

here s is the arc length along the ray path

- Several realized ray pathes have the same optical path length

$$L = \int_{P_1}^{P_2} n \cdot \vec{s} \cdot d\vec{r} = \text{const.}$$

- The principle is valid for smooth and discrete index distributions





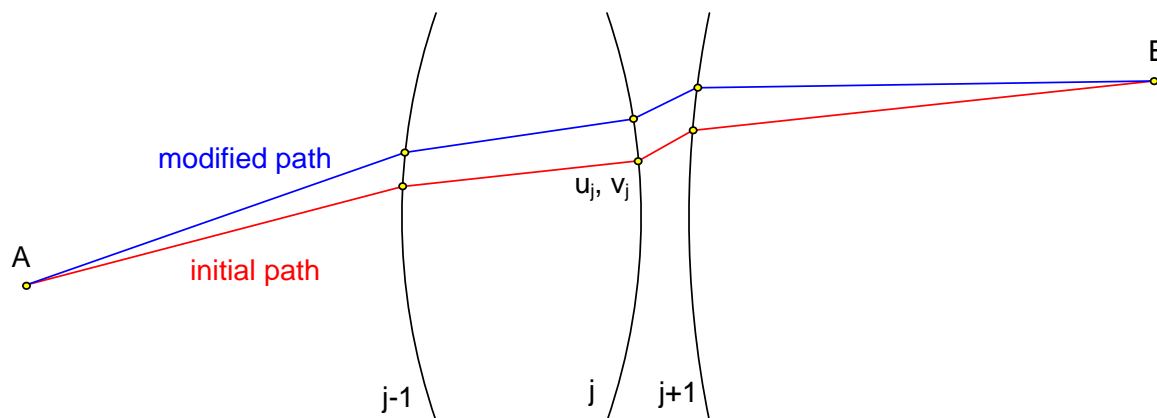
Fermat Principle

- The Fermat principle states, that a modified ray path must have a larger optical path length
- The realized path is a minimum and therefore the first derivatives to path variables must vanish

$$\frac{\partial L(u, v)}{\partial u} = 0, \quad \frac{\partial L(u, v)}{\partial v} = 0$$

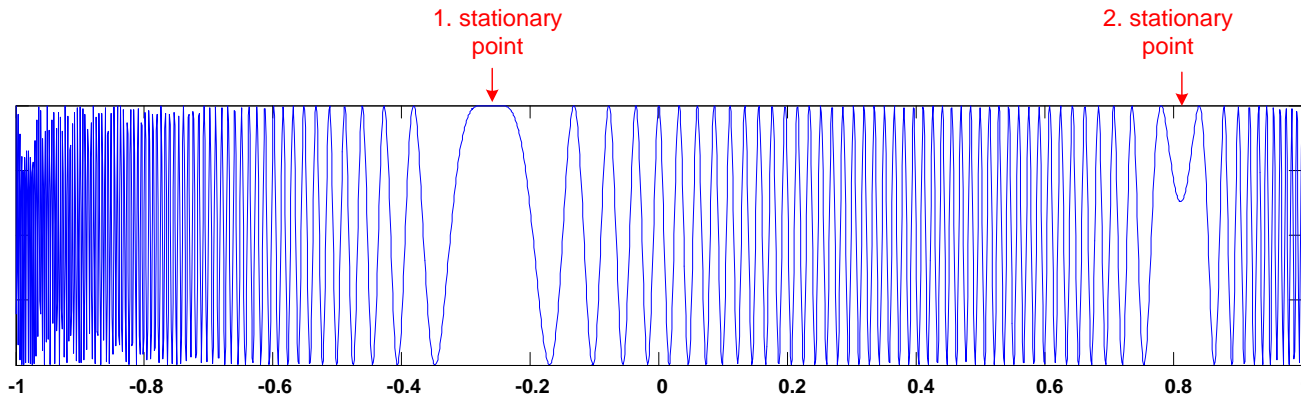
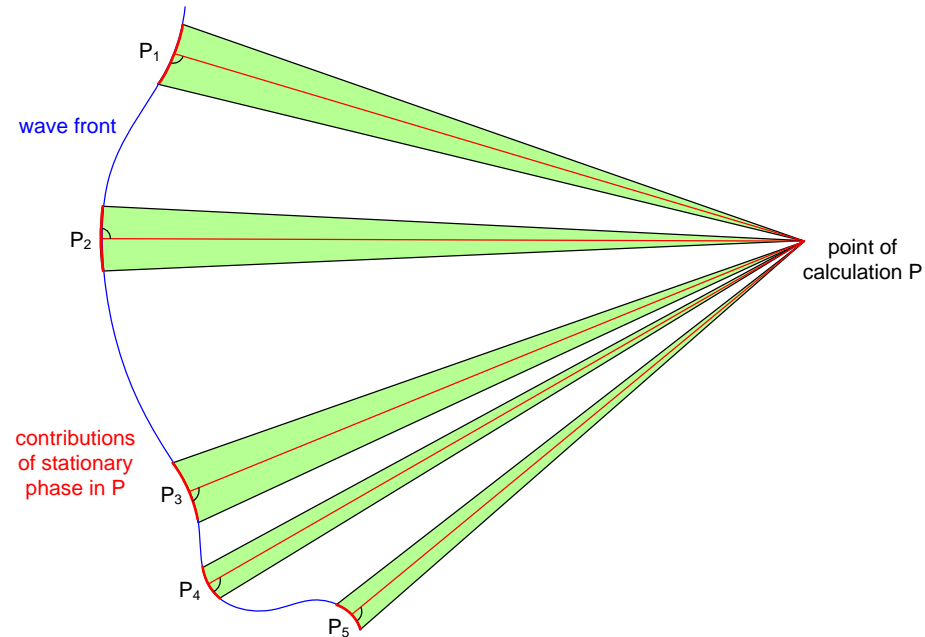
u and v are arbitrary coordinates on the surface, which indicate the ray intersection point

- The phase has a stationary point with minimal optical path length
- In the special case of conjugated points A and B, this is true for every ray in a perfect system



Principle of Stationary Phase

- Principle of stationary phase from an illustrative point of view
- Oscillatory parts are cancelled out
- The light delivers constructive interference in those directions, which has stationary phase contributions: the ray direction perpendicular to the wave
- Critical are stationary phase contributions of boundary points





Principle of Stationary Phase

- Principle of stationary phase from wave optical viewpoint
- Diffraction integral as plane wave decomposition with phase and slowly varying amplitude

$$E(x, y, z) = \iint E(s_x, s_y) \cdot e^{\frac{2\pi i}{\lambda} (xs_x + ys_y + zs_z)} ds_x ds_y$$

$$E(x, y, z) = \iint A(s_x, s_y) \cdot e^{\frac{2\pi i}{\lambda} L(s_x, s_y)} \cdot e^{\frac{2\pi i}{\lambda} [xs_x + ys_y + zs_z]} ds_x ds_y$$

- Oscillatory contributions cancel out except the point of stationary phase

$$\frac{d}{ds_x} [L(s_x, s_y) + xs_x + ys_y + zs_z] = 0 \quad , \quad \frac{d}{ds_y} [L(s_x, s_y) + xs_x + ys_y + zs_z] = 0$$

- This gives the solutions

$$x = -\frac{dL}{ds_x} + \frac{s_x}{s_z} \cdot z \quad , \quad y = -\frac{dL}{ds_y} + \frac{s_y}{s_z} \cdot z$$

with the normalization relation $s_x^2 + s_y^2 + s_z^2 = 1$

- The light propagates along the ray path perpendicular to the phase / wave front



Law of Malus-Dupin

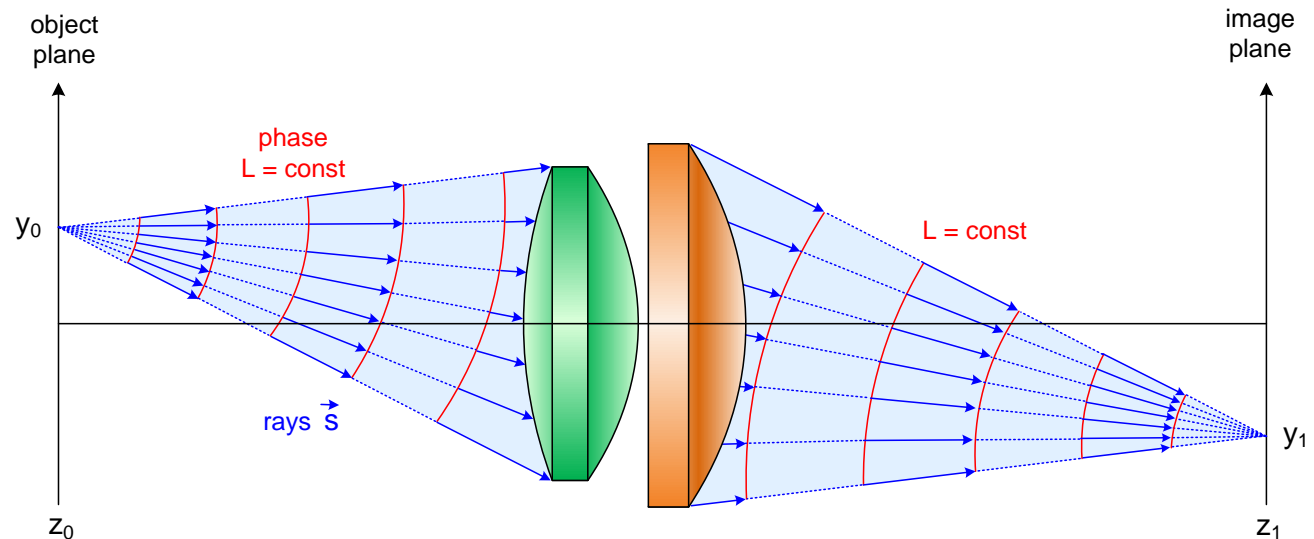
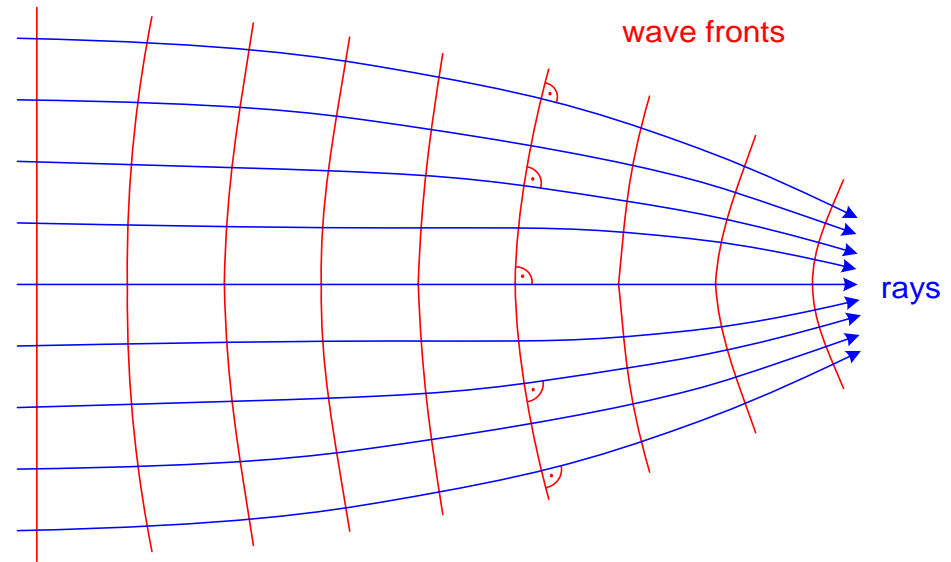
- Law of Malus-Dupin:
 - equivalence of rays and wavefronts
 - both are orthonormal
 - identical information

- Condition:
No caustic of rays

- Mathematical:
Rotation of Eikonal
vanish

$$\text{rot}(n \cdot \vec{s}) = 0$$

- Optical system:
Rays and spherical
waves orthonormal

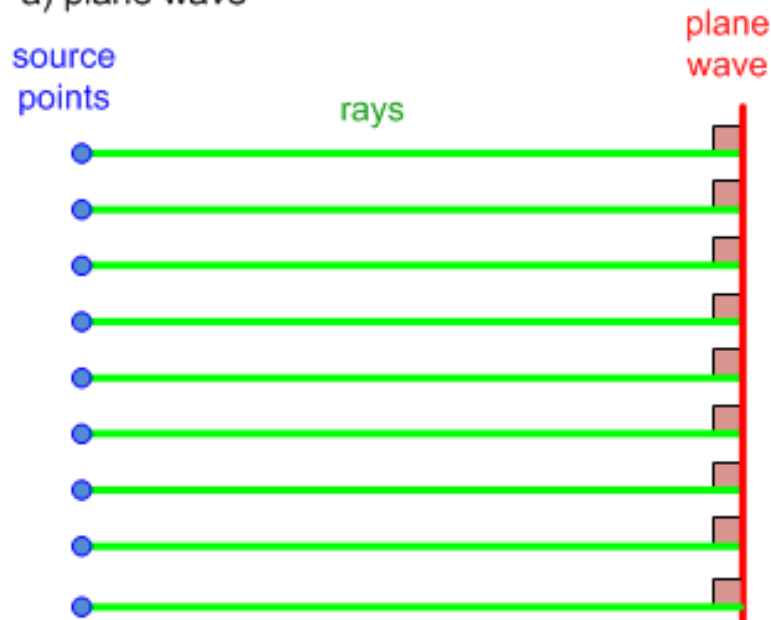




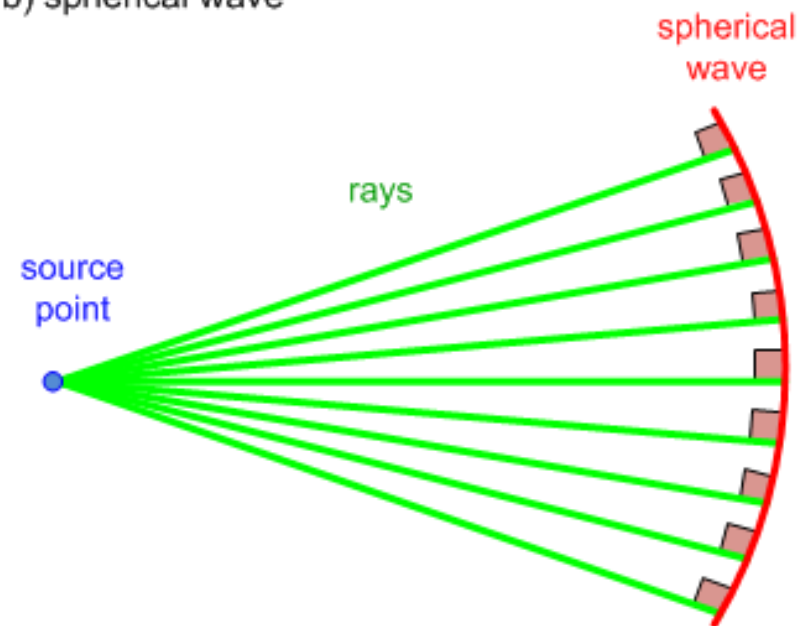
Ray-Wave Equivalent

- Rays and waves carry the same information
- Wave surface is perpendicular on the rays
- Wave is purely geometrical and has no diffraction properties

a) plane wave



b) spherical wave





Lagrange and Hamilton Formulation

- The point eikonal or optical path length serves as the Lagrange function written in coordinate representation with derivative with respect to z gives the optical Lagrange formulation
- With the definition of the impulse variables
- The equation of motion reads in vectorial notation corresponds to the eikonal equation
- The Hamilton version is given by the Legendre transform

$$L = \int n(\vec{r}) d\vec{r}$$

$$\begin{aligned} ds &= |d\vec{r}| = \sqrt{dx^2 + dy^2 + dz^2} \\ &= dz \cdot \sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} = dz \cdot \sqrt{1 + \dot{x}^2 + \dot{y}^2} \end{aligned}$$

$$L(x, y, \dot{x}, \dot{y}, z) = n(x, y, z) \cdot \sqrt{1 + \dot{x}^2 + \dot{y}^2}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = n \cdot \frac{\dot{x}}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}} = n \cdot \frac{dx}{ds} = n \cdot s_x$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = n \cdot \frac{\dot{y}}{\sqrt{1 + \dot{x}^2 + \dot{y}^2}} = n \cdot \frac{dy}{ds} = n \cdot s_y$$

$$\frac{d}{ds} \left(n \cdot \frac{d\vec{r}}{ds} \right) = \nabla n \quad n(\vec{r}) = |\nabla L|$$

$$\begin{aligned} H(x, y, p_x, p_y) &= p_x \cdot x' + p_y \cdot y' - L \\ &= -\sqrt{n^2 - p_x^2 - p_y^2} \end{aligned}$$

Eikonal Formulation of Imaging

- A ray is described in a plane of constant z by 4 variables
 - point x, y
 - direction or angles s_x, s_y

- An optical system transmits an initial ray into a ray in the image space

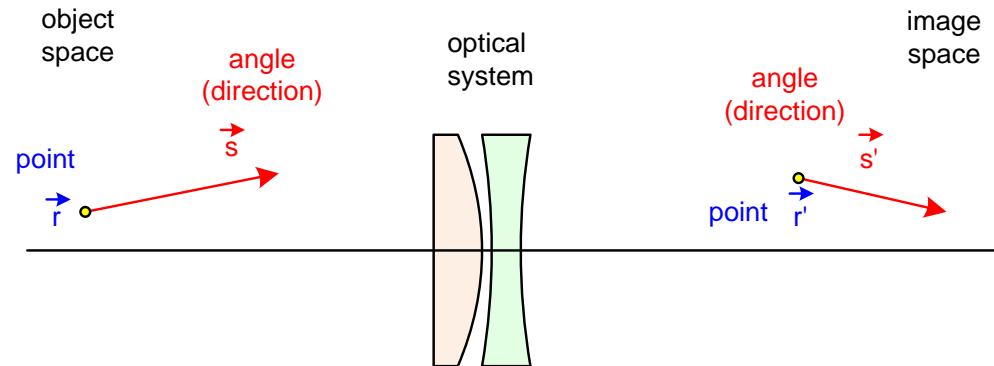
- In the most general mathematical description, 4 functions control this ray transform

$$\begin{aligned} x' &= L_1(x, y, s_x, s_y) & , & & y' &= L_2(x, y, s_x, s_y) \\ s_x' &= L_3(x, y, s_x, s_y) & , & & s_y' &= L_4(x, y, s_x, s_y) \end{aligned}$$

- The Fermat principle restricts this most general approach to one single function, the so called eikonal function L

$$(x', y', s_x', s_y') = L(x, y, s_x, s_y)$$

- In reality, there are only 4 degrees of freedom:
for a pre-given initial ray, the transferred ray is fixed





Eikonal Formulation of Imaging

- There are 4 possible options to formulate the problem concerning the choice of the independent variables combining object and image space

object space		image space				
point	angle	point	angle	eikonal	inventor	pitfall
x, y		x', y'		point	Hamilton, Bruns	conjugated
	s_x, s_y		s'_x, s'_y	angle	Schwarzschild, Seidel	telescopic
x, y			s'_x, s'_y	point-angle		afocal, infinity image
	s_x, s_y	x', y'		angle-point		afocal, infinity object

- The different eikonal functions can be calculated via a variable transform by a Legendre transform
- The eikonal functions have singularities, which limits their application
Example: if conjugated planes are considered, x', y' are fully determined by x, y .
Therefore there are no 4 independent variables and the point eikonal fails
- There is not one description, which is valid and useful for all possible cases

Point Eikonal

- Point Eikonal $L(x, y, x', y')$:

Optical path length from point $P(x, y)$ in object space to point $P'(x', y')$ in image space

- Total differential of the Hamilton Eikonal for purely transverse directions and $z = z' = 0$

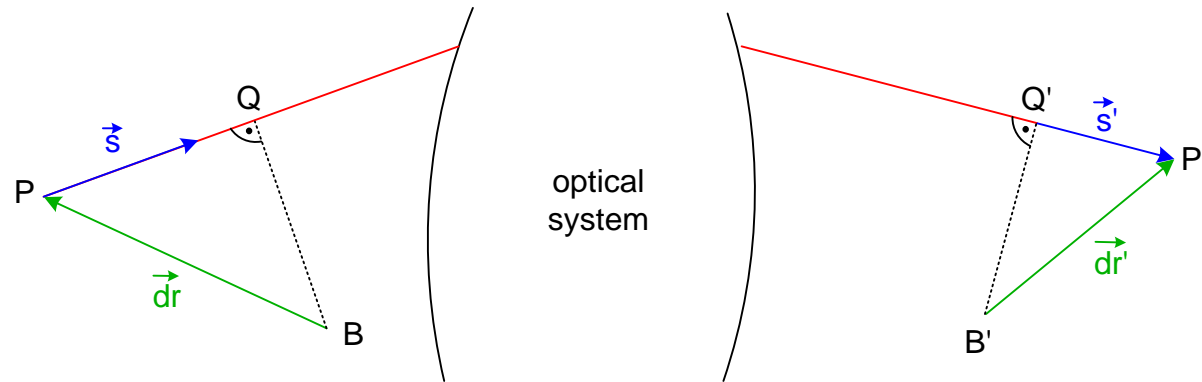
$$dL_P(x, y, x', y') = n' \cdot (s'_x dx' + s'_y dy') - n \cdot (s_x dx + s_y dy)$$

$$dL_P(\vec{r}, \vec{r}') = n' \cdot d\vec{r} \cdot \vec{s} - n \cdot d\vec{r}' \cdot \vec{s}'$$

- Differential equations

$$\frac{\partial L_P}{\partial x} = -ns_x \quad \frac{\partial L_P}{\partial y} = -ns_y$$

$$\frac{\partial L_P}{\partial x'} = n's'_x \quad \frac{\partial L_P}{\partial y'} = n's'_y$$



- Physical interpretation:

Given point in object and

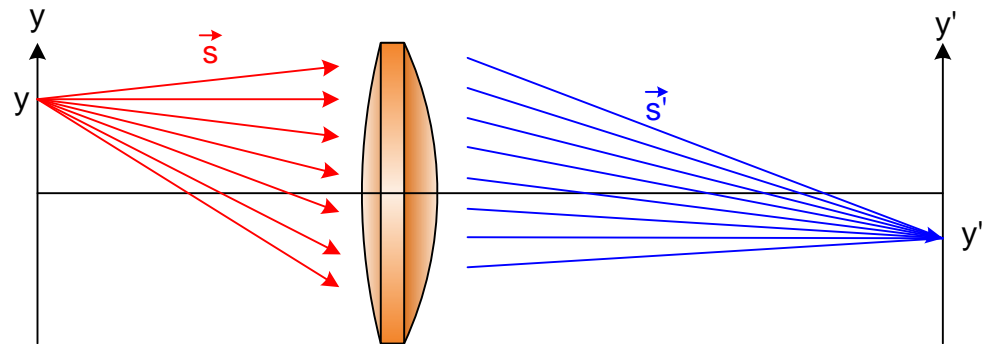
image space:

Integration of the equations

give the corresponding directions

- Not valid in conjugated planes:

x', y' does not depend on path s, s'



Point-Angle Eikonal

- Legendre-transformation of point eikonal:
point angle eikonal

$$L_{PA}(x, y, s'_x, s'_y) = L_P(x, y, x', y') - n' \cdot (x' s'_x + y' s'_y)$$

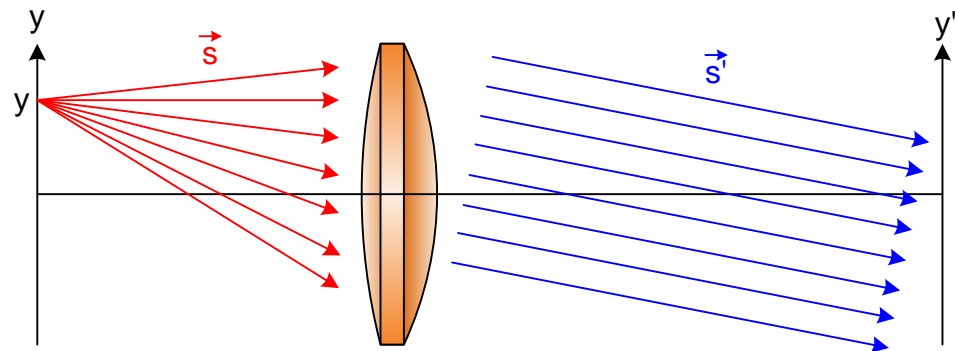
- Total differential

$$dL_{PA}(x, y, s'_x, s'_y) = -n' \cdot (x' ds'_x + y' ds'_y) - n \cdot (s_x dx + s_y dy)$$

- Derivatives:

$$\frac{\partial L_{PA}}{\partial x} = -ns_x \quad \frac{\partial L_{PA}}{\partial y} = -ns_y \quad \frac{\partial L_{PA}}{\partial s'_x} = -n'x' \quad \frac{\partial L_{PA}}{\partial s'_y} = -n'y'$$

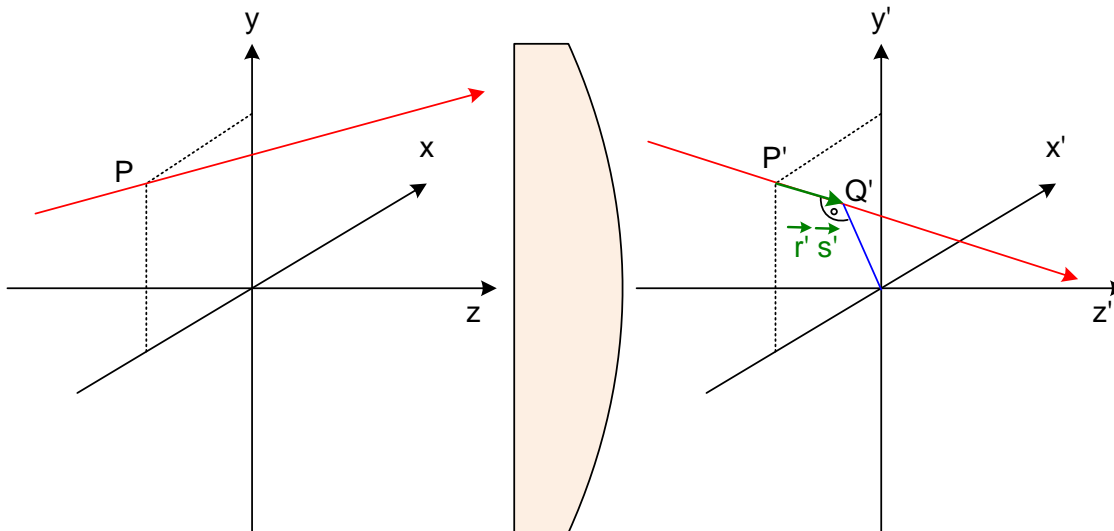
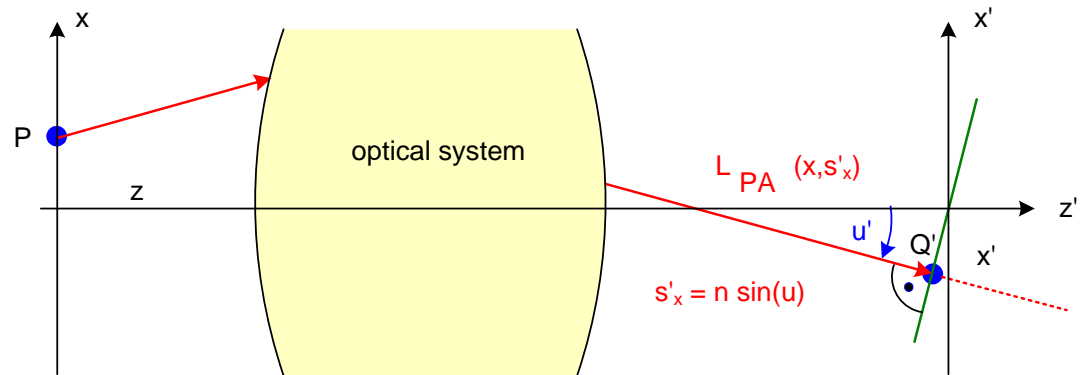
- L_{PA} defines a ray with starting point and final direction
- The point angle eikonal can be used for conjugated planes
- Point angle eikonal is not applicable for an afocal system:
 s'_x, s'_y are independent of x, y





Point-Angle Eikonal

- Interpretation of the point angle eikonal
- Optical path length from the starting point P to the point Q' in the image space, which corresponds to the perpendicular projection



- Corresponding Legendre transform gives the pure angle eikonal according to Schwarzschild

$$dL_A(s_x, s_y, s_x', s_y') = -n' \cdot (x' ds_x' + y' ds_y') + n \cdot (x ds_x + y ds_y)$$

- Vectorial formulation

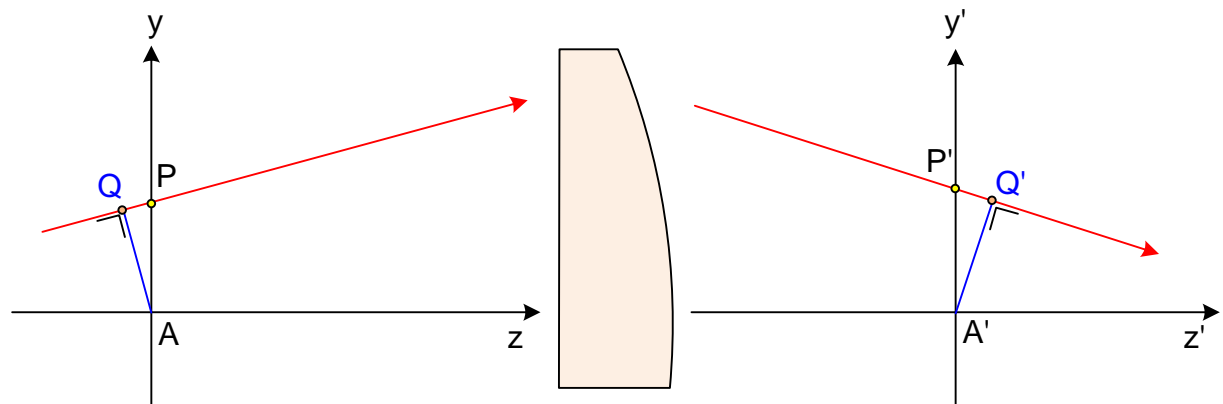
$$dL_A(\vec{s}, \vec{s}') = n \cdot (\vec{r} - \vec{a}) \cdot d\vec{s} - n' \cdot (\vec{r}' - \vec{a}') \cdot d\vec{s}'$$

- The ray is defined by its directions only

- Differential equations

$$\frac{\partial L_A}{\partial s_x} = nx \quad \frac{\partial L_A}{\partial s_y} = ny \quad \frac{\partial L_A}{\partial s_x'} = -n'x' \quad \frac{\partial L_A}{\partial s_y'} = -n'y'$$

- Interpretation:
optical path length between the feet points Q, Q'
perpendicular to the ray





Reciprocity Relation

- Point angle eikonal

$$\frac{\partial L_{AP}}{\partial s_x} = nx$$

$$\frac{\partial L_{AP}}{\partial x'} = n' \cdot s'_x$$

- Independence of mixed derivatives

$$\frac{\partial^2 L_{AP}}{\partial s_x \partial x} = n$$

$$\frac{\partial^2 L_{AP}}{\partial x' \partial s'_x} = n'$$

- Result:

- reciprocity relation

$$n \cdot \left(\frac{\partial s_x}{\partial s'_x} \right) = n' \cdot \left(\frac{\partial x'}{\partial x} \right)$$

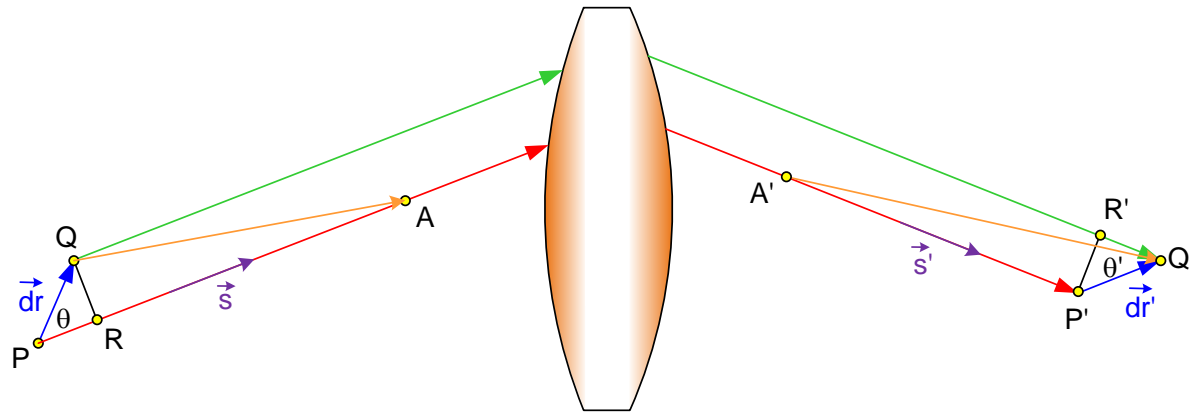
- relation between lateral and angle magnification

$$n \cdot m_A = n' \cdot m$$



Optical Path Length of an Optical System

- A ray from P goes through a system to P'
- Change of initial point P to Q
- Comparison of optical path of P and Q via A:
Difference: Hamilton eikonal



$$\begin{aligned}\delta L &= \overline{QQ'} - \overline{PP'} = \overline{QAA'Q'} - \overline{PAA'P'} = \overline{P'R'} - \overline{PR} \\ &= n' dr' \cos \theta' - n dr \cos \theta \\ &= n' \vec{s}' \cdot d\vec{r}' - n \vec{s} \cdot d\vec{r}\end{aligned}$$

- Total differential: $\delta L_{Ham} = n' (dx' s'_x + dy' s'_y + dz' s'_z) - n (dx s_x + dy s_y + dz s_z)$

- spatial point eikonal
- Differential equations

$$\frac{\partial L_{Ham}}{\partial x} = -n s_x \quad \frac{\partial L_{Ham}}{\partial y} = -n s_y \quad \frac{\partial L_{Ham}}{\partial z} = -n s_z$$

$$\frac{\partial L_{Ham}}{\partial x'} = n' s'_x \quad \frac{\partial L_{Ham}}{\partial y'} = n' s'_y \quad \frac{\partial L_{Ham}}{\partial z'} = n' s'_z$$

- Change of initial point:
the change in the final point is fixed by the eikonal



Paraxial Point Eikonal for a Refracting Surface

- Refracted ray by a spherical dielectric interface
path difference

$$L(r) = n \cdot \sqrt{(z-s)^2 + (r-x)^2} + n' \cdot \sqrt{(-z+s')^2 + (r-x')^2}$$

- Paraxial approximation:
Taylor expansion for small x, x', r, z

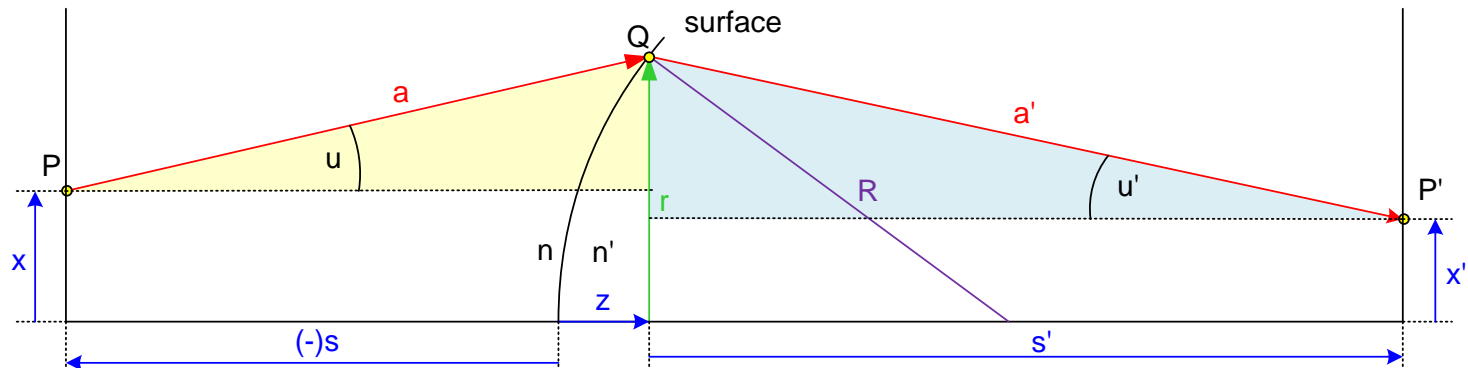
$$\begin{aligned} L(r) &= \left[-ns + n's' - \frac{nx^2}{2s} + \frac{n'x'^2}{2s'} \right] + r \cdot \left[+\frac{nx}{s} - \frac{n'x'}{s'} \right] \\ &\quad + r^2 \cdot \frac{1}{2} \left[-\frac{n}{s} + \frac{n'}{s'} - \frac{n'-n}{R} \right] \\ &= A + Br + Cr^2 \end{aligned}$$

- Stationary phase condition

$$\frac{dL(r)}{dr} = B + 2r \cdot C = 0$$

- Angles u, u'

$$u = \frac{r-x}{-s}, \quad u' = \frac{r-x'}{s'}$$





Paraxial Point Eikonal for a Refracting Surface

- Paraxial imaging
conditions for $B = 0$ and $C = 0$

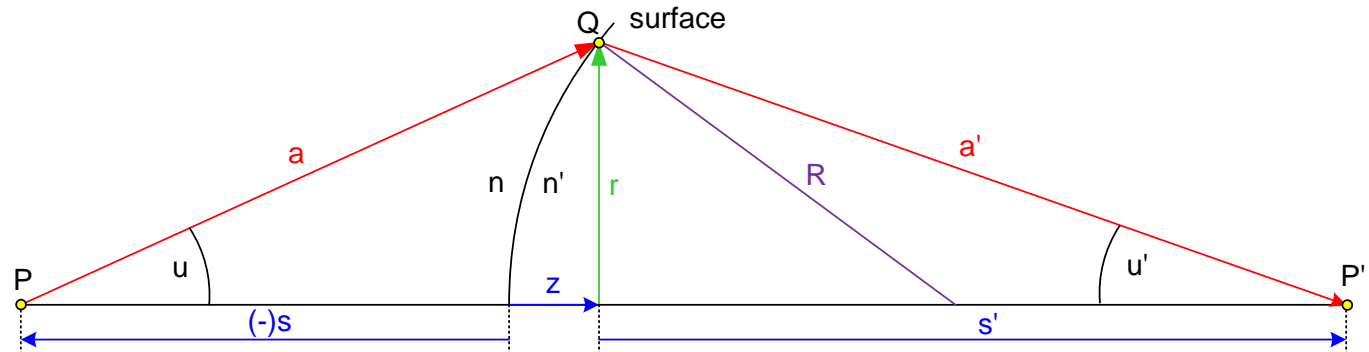
$$+\frac{nx}{s} - \frac{n'x'}{s'} = 0 \quad , \quad -\frac{n}{s} + \frac{n'}{s'} - \frac{n'-n}{R} = 0$$

1. Lens makers formula

$$-\frac{n}{s} + \frac{n'}{s'} = \frac{n'-n}{R}$$

2. definition of magnification

$$m = \frac{x'}{x} = \frac{n \cdot s'}{n' \cdot s}$$





Point-Angle Eikonal and Lens Aberrations

- The point angle eikonal of a lens is suitable to describe the aberrations

$$dL_{PA}(x, y, s_x', s_y') = -n' \cdot (x' ds_x' + y' ds_y') - n \cdot (s_x dx + s_y dy)$$

- The derivatives give the deviations:
 1. of the position:
transverse aberrations as deviation from perfect paraxial location

$$\frac{\partial L_{PA}}{\partial s_x'} = -n' x' \quad \frac{\partial L_{PA}}{\partial s_y'} = -n' y'$$

$$x' = m \cdot x + \Delta x' = -\frac{1}{n'} \cdot \frac{\partial L_{PA}}{\partial s_x'} \quad , \quad \Delta x' = -m \cdot x - \frac{1}{n'} \cdot \frac{\partial L_{PA}}{\partial s_x'}$$

$$y' = m \cdot y + \Delta y' = -\frac{1}{n'} \cdot \frac{\partial L_{PA}}{\partial s_y'} \quad , \quad \Delta y' = -m \cdot y - \frac{1}{n'} \cdot \frac{\partial L_{PA}}{\partial s_y'}$$

2. of the direction:
angle aberrations correspondingly

$$\frac{\partial L_{PA}}{\partial x} = -n s_x \quad \frac{\partial L_{PA}}{\partial y} = -n s_y$$

- The mixed derivative of the eikonal gives a relationship between the transverse aberration components



Angle Eikonal for a Refracting Surface

- P point on ray, A arbitrary point on axis

- Real surface

$$z = \frac{x^2 + y^2}{2R} + \frac{(x^2 + y^2)^2}{8R^3} \cdot (1+b) + \dots$$

- Angle eikonal

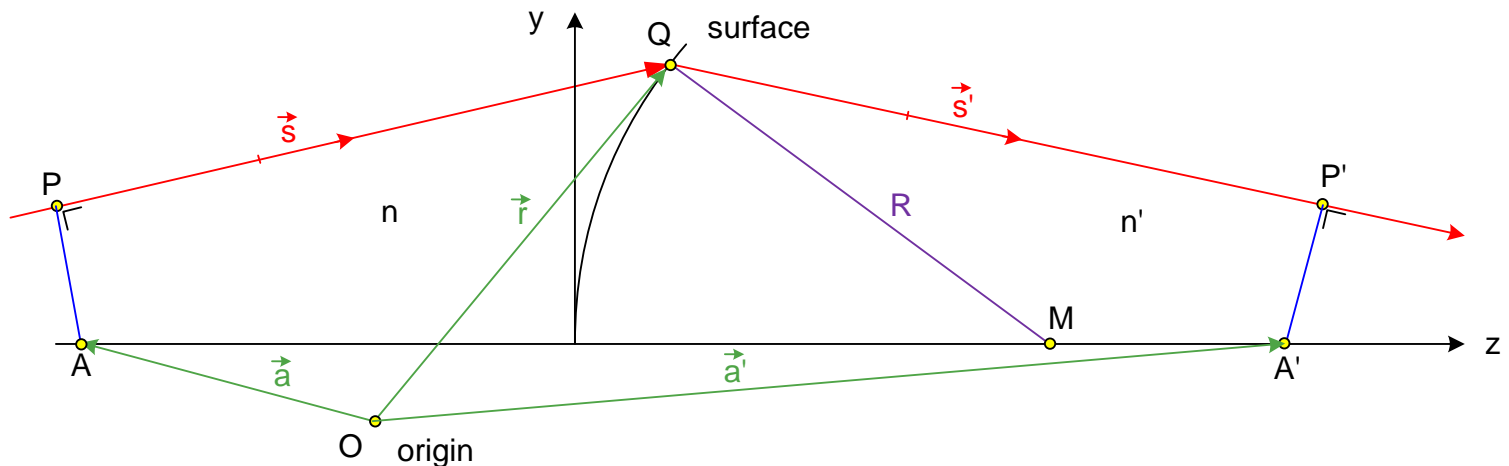
$$dL_A(\vec{s}, \vec{s}') = n \cdot (\vec{r} - \vec{a}) \cdot d\vec{s} - n' \cdot (\vec{r}' - \vec{a}') \cdot d\vec{s}'$$

- In coordinate representation

with

$$dL_A = n \cdot [x \cdot ds_x + y \cdot ds_y + (z - a) \cdot ds_z] - n' \cdot [x \cdot ds'_x + y \cdot ds'_y + (z - a') \cdot ds'_z]$$

$$s_z = \sqrt{1 - s_x^2 - s_y^2}, \quad s'_z = \sqrt{1 - s'^2_x - s'^2_y}$$





Angle Eikonal for a Refracting Surface

- Result for 4th order Taylor approximation (x, y are still parameters, which should be eliminated)

$$\begin{aligned}
 dL_A = & -n \cdot a + n' \cdot a' \\
 & + n \cdot \left[x \cdot s_x + y \cdot s_y + \frac{x^2 + y^2}{2R} + \frac{a}{2} \cdot (s_x^2 + s_y^2) \right] \\
 & - n' \cdot \left[x \cdot s'_x + y \cdot s'_y + \frac{x^2 + y^2}{2R} + \frac{a'}{2} \cdot (s'^2_x + s'^2_y) \right] \\
 & + n \cdot \left[\frac{1+b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s_x^2 + s_y^2)}{4R} + \frac{a}{8} \cdot (s_x^2 + s_y^2)^2 \right] \\
 & - n' \cdot \left[\frac{1+b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s'^2_x + s'^2_y)}{4R} + \frac{a'}{8} \cdot (s'^2_x + s'^2_y)^2 \right]
 \end{aligned}$$

- Use of refraction law to eliminate x, y

$$dL_A^{(4)} = S_1 \cdot u^2 + S_2 \cdot v^2 + S_3 \cdot w^2 + S_4 \cdot u \cdot v + S_5 \cdot u \cdot w + S_6 \cdot v \cdot w$$

with the rotational invariants

$$u = s_x^2 + s_y^2, \quad v = s'^2_x + s'^2_y, \quad w = s_x s'_x + s_y s'_y$$

and coefficients S_j



Angle Eikonal for a Refracting Surface

▪ Coefficients S_j

Description of the optical path as
a function of

1. system data: n , n' , R , b
2. ray parameter a , a'

$$S_1 = \frac{a}{8n^3} - \frac{R}{4(n-n')^2} \cdot \left[\frac{1}{n} + \frac{1+b}{2(n'-n)} \right]$$

$$S_2 = -\frac{a'}{8n'^3} - \frac{R}{4(n-n')^2} \cdot \left[-\frac{1}{n'} + \frac{1+b}{2(n'-n)} \right]$$

$$S_3 = -\frac{(1+b)R}{2(n'-n)^3}$$

$$S_4 = -\frac{R}{4(n-n')^2} \cdot \left[\frac{1}{n} - \frac{1}{n'} + \frac{1+b}{n'-n} \right]$$

$$S_5 = \frac{R}{2(n-n')^2} \cdot \left[\frac{1}{n} + \frac{1+b}{n'-n} \right]$$

$$S_6 = \frac{R}{2(n-n')^2} \cdot \left[-\frac{1}{n} + \frac{1+b}{n'-n} \right]$$



Perfect Imaging

- Eikonal theory:
 - perturbation method
 - zero order is the paraxial approximation
 - higher order perturbation corresponds to aberrations

- Perfect imaging as special cases:
 1. Stigmatic imaging with conic sections finite-finite (ellipsoid), infinite-fine (parabola)
 2. Special refractive index distributions, e.g. Maxwellian fish-eye
 3. For infinitesimal field size and finite aperture cone: aplanatic imaging condition

- Imperfect lens: magnification and transverse aberrations

$$x' = m \cdot x + \Delta x \quad , \quad y' = m \cdot y + \Delta y$$

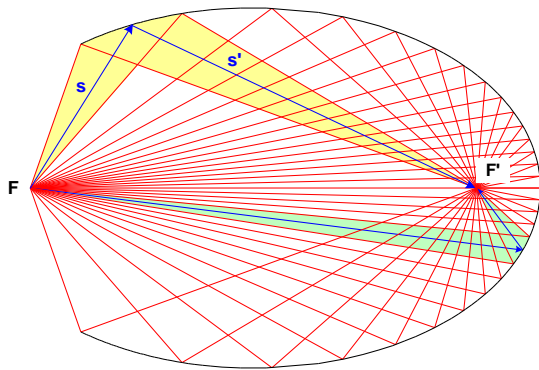
- From point angle eikonal differential equations:

$$x' = -\frac{1}{n'} \cdot \frac{\partial L_{PA}(x, y, s'_x, s'_y)}{\partial s'_x} \quad , \quad y' = -\frac{1}{n'} \cdot \frac{\partial L_{PA}(x, y, s'_x, s'_y)}{\partial s'_y}$$

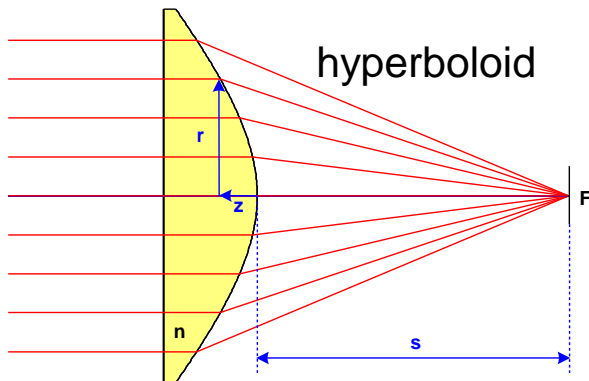
- Perfect imaging as special cases:

Examples

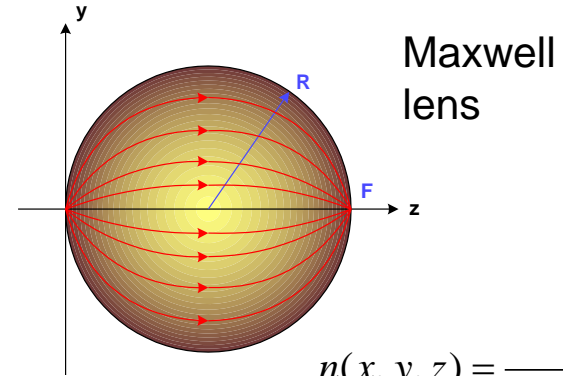
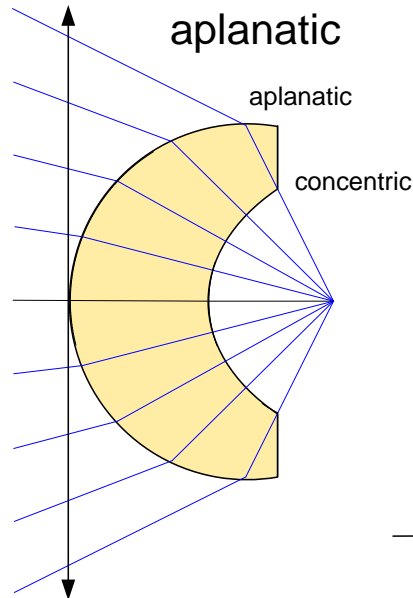
ellipsoid



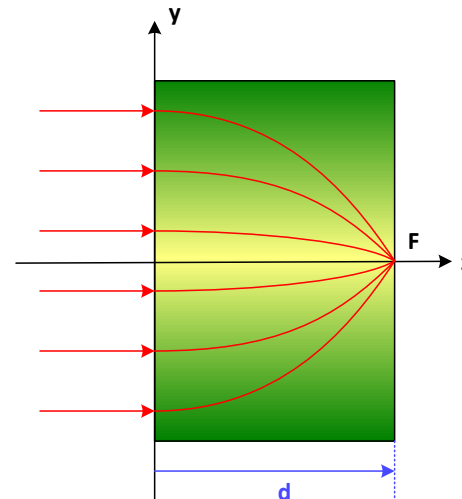
hyperboloid



aplanatic



$$n(x, y, z) = \frac{2n_{env}}{1 + \frac{x^2 + y^2 + z^2}{R^2}}$$



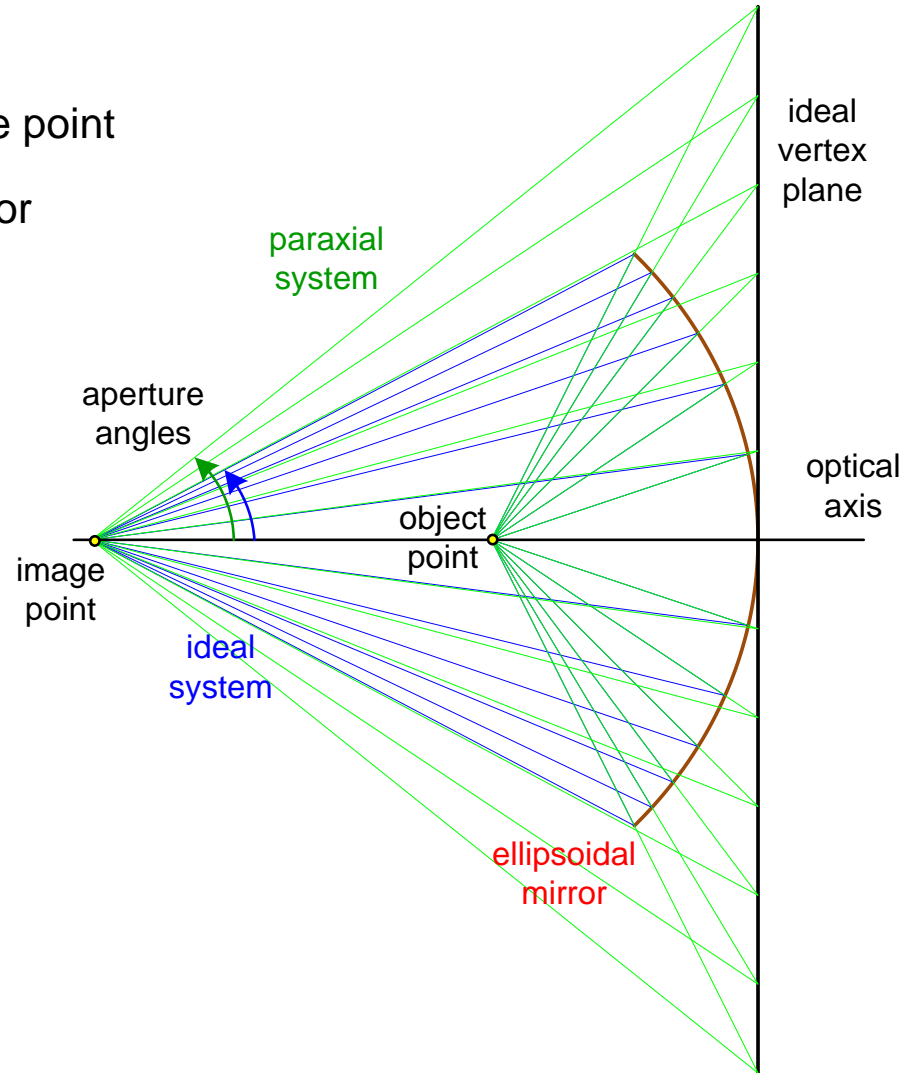
Mikhaelian
lens

$$n(r) = \frac{n_0}{\cosh\left(\frac{\pi}{2d} \cdot r\right)}$$

What is ,Ideal' ?



- The notation ,ideal' imaging is not unique
- Ideal is in any case the location of the image point
- The geometrical ray paths can be different for
 1. paraxial
 2. ideal / linear collineation
 3. aplanatic
- The photometric properties are different due to non-equidistant sampling
- If a perfect lens is idealized in a software as one surface, there are principal discrepancies in the location of the intersection points



Abbe Sine Condition

- If for example a small field area and a widespread ray bundle is considered, a perfect imaging is possible

- The eikonal with the expression
can be written for $\delta L=0$ as

$$\delta L = n' \vec{s}' \cdot d\vec{r}' - n \vec{s} \cdot d\vec{r}$$

$$n \cdot \vec{s} \cdot d\vec{r} = n' \cdot \vec{s}' \cdot d\vec{r}'$$

$$n \cdot dr \cdot \cos \theta = n' \cdot dr' \cdot \cos \theta'$$

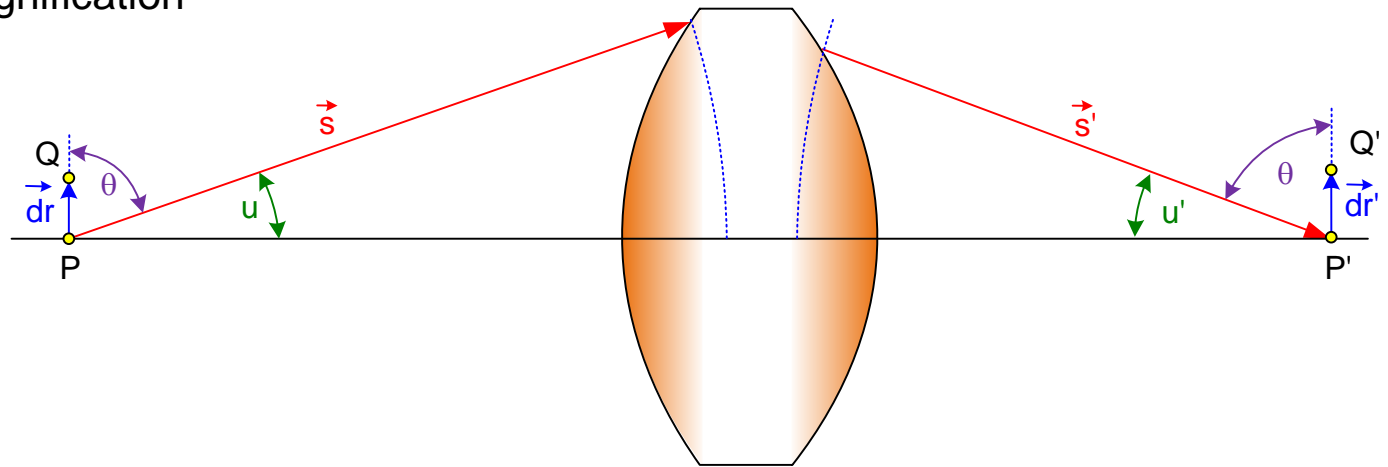
$$n \cdot \cos \theta = n' \cdot m \cdot \cos \theta'$$

- In the special case of an angle 90° we get with $\cos(\theta)=\sin(u)$ the Abbe sine condition

$$m = \frac{n \sin u}{n' \sin u'} = \frac{y'}{y}$$

with the lateral magnification

$$m = \frac{d\vec{r}'}{d\vec{r}}$$





Ray-Wave Relationships

- Concrete calculation of wave aberration:
addition of discrete optical path lengths
(OPL)
- Reference on chief ray and reference
sphere (optical path difference)
- Relation to transverse aberrations
- Conversion between longitudinal
transverse and wave aberrations
- Scaling of the phase / wave aberration:
 1. Phase angle in radian
 2. Light path (OPL) in mm
 3. Light path scaled in λ

$$l_{OPL} = \int_{OE}^{AP} n \cdot d\vec{r}$$

$$\Delta_{OPD}(x, y) = l_{OPL}(x, y) - l_{OPL}(0, 0)$$

$$\frac{\partial W}{\partial y_p} = -\frac{\Delta y'}{R - W} \approx -\frac{\Delta y'}{R}$$

$$\Delta s' = \frac{R}{y_p} \cdot \Delta y' = \frac{\Delta y'}{\sin u'} = \frac{R^2}{y_p} \cdot \frac{\partial W(x_p, y_p)}{\partial y_p}$$

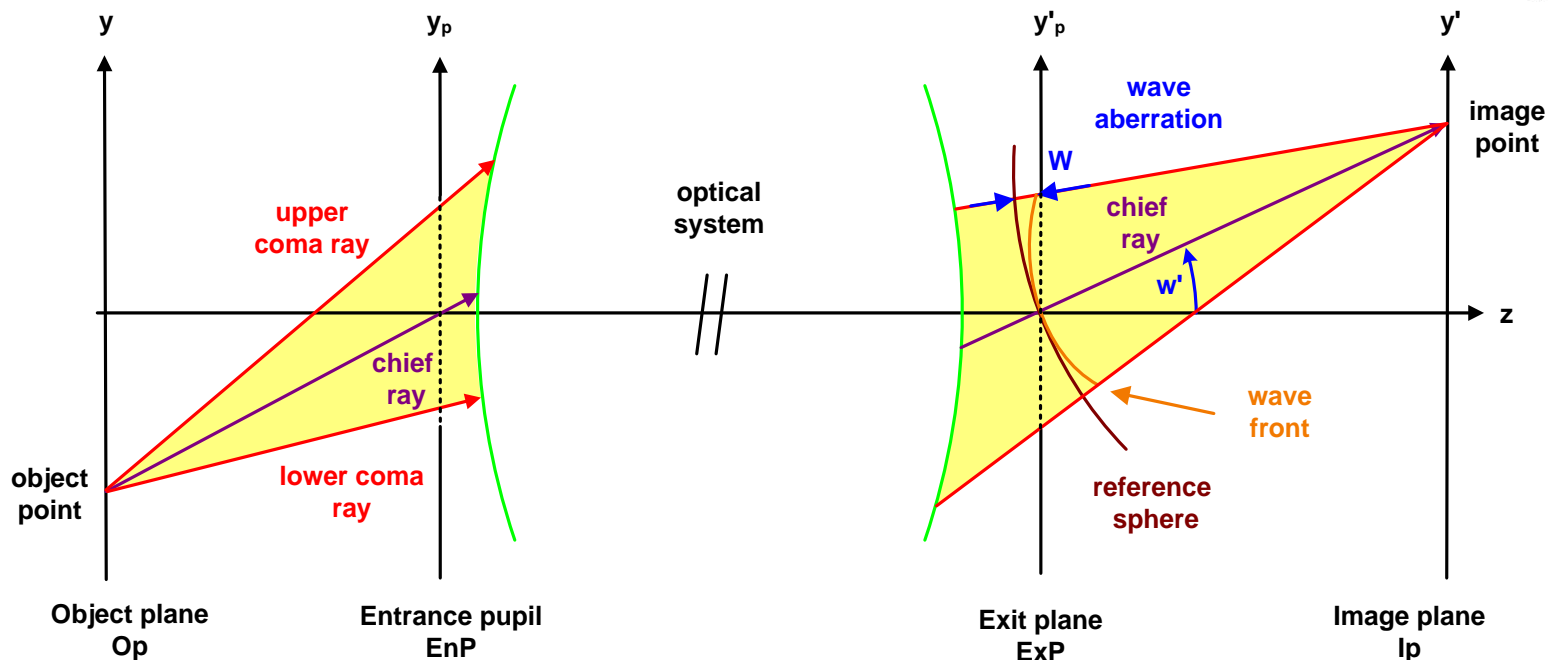
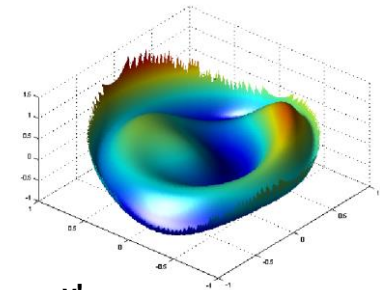
$$E(x) = A(x) \cdot e^{i \cdot \varphi(x)}$$

$$E(x) = A(x) \cdot e^{i \cdot k \Delta_{OPD}(x)}$$

$$E(x) = A(x) \cdot e^{2\pi i \cdot W(x)}$$

Wave Aberration in Optical Systems

- Definition of optical path length in an optical system:
Reference sphere around the ideal image point through the center of the pupil
- Chief ray serves as reference
Difference of OPL : optical path difference OPD
- Practical calculation: discrete sampling of the pupil area,
real wave surface represented as matrix





Wave Aberration

- Exact relation between wave aberration and ray deviation
- General expression from geometry describes the lateral aberration
- Substitution of angle by scalar product
- Exact relation is quadratic in R
- Approximation for large R

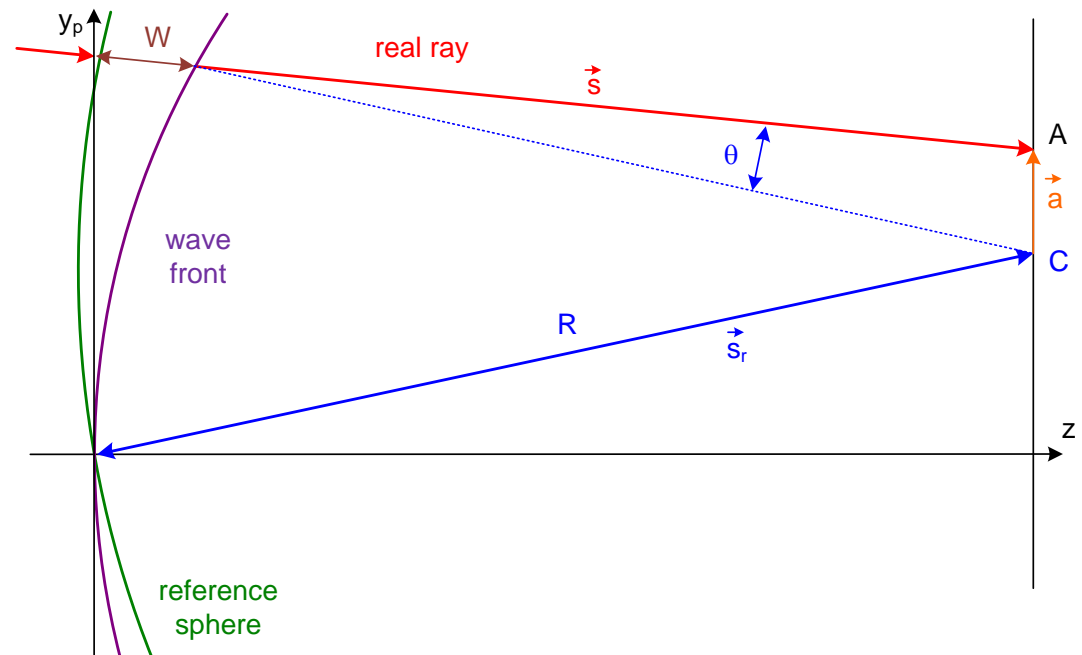
$$W = W_0 + n' \cdot \frac{[\vec{s}_r - \vec{s} \cdot (\vec{s} \cdot \vec{s}_r)] \cdot \vec{a}}{1 + \vec{s} \cdot \vec{s}_r} + n' \cdot \frac{a^2 - (\vec{s} \cdot \vec{a})^2}{R \cdot (1 + \cos \theta)}$$

$$W = W_\infty + n' \cdot \frac{a^2 - (\vec{s} \cdot \vec{a})^2}{2R} \cdot \left[1 + \frac{a^2 - (\vec{s} \cdot \vec{a})^2}{4R^2} \right]$$

$$\Delta W = -\frac{x_p}{R} \cdot \Delta x' - \frac{y_p}{R} \cdot \Delta y' - \frac{x_p^2 + y_p^2}{2 \cdot R} \cdot \Delta z'$$

with

$$\vec{a} = \begin{pmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}$$



Conversion Ray - Wave

- Rays and wavefronts are equivalent

$$E(x, y) = A(x, y) \cdot e^{2\pi i W(x, y)}$$

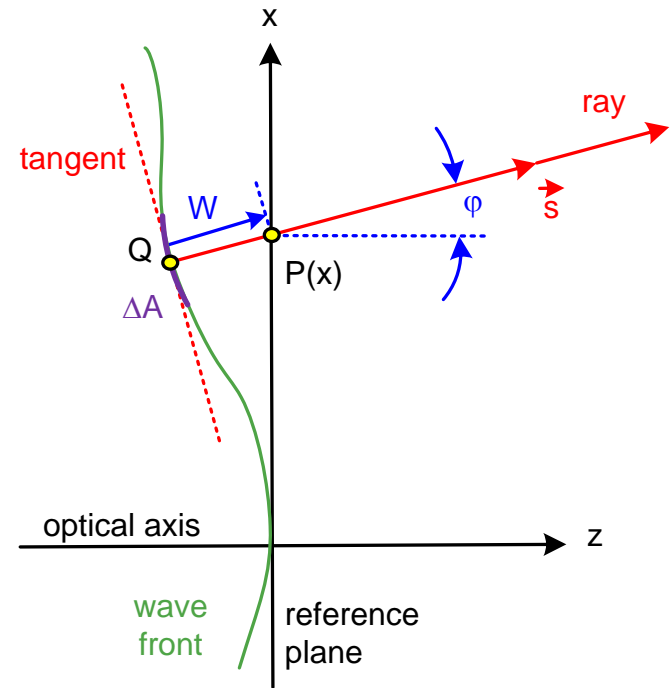
- Phase corresponds to ray direction

$$\vec{s} = \lambda \cdot \begin{pmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \sqrt{\left(\frac{1}{\lambda}\right)^2 - \left(\frac{\partial W}{\partial x}\right)^2 - \left(\frac{\partial W}{\partial y}\right)^2} \end{pmatrix}$$

$$\frac{\partial W}{\partial x} = \frac{1}{\lambda} \cdot s_x, \quad \frac{\partial W}{\partial y} = \frac{1}{\lambda} \cdot s_y$$

- Amplitude A is described by ray weighting factor g, Transform of area element by Jacobian

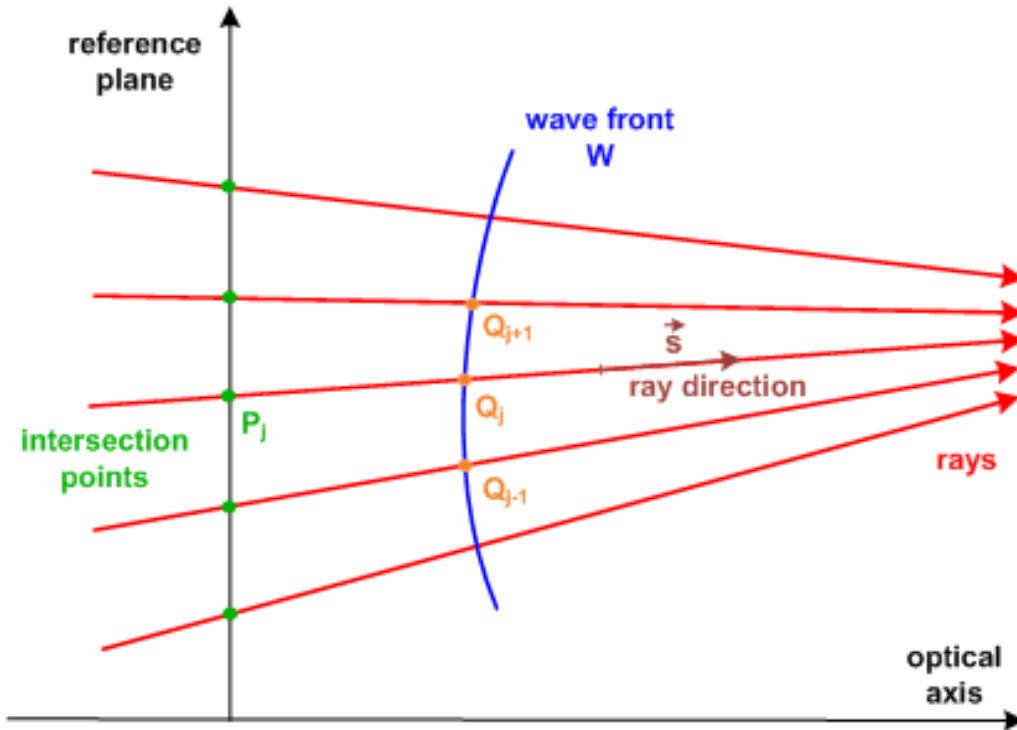
$$g = \Delta A^2 \quad \Delta A' = \frac{\Delta A}{\begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial x} \end{vmatrix}}$$





Conversion Ray - Wave

- Realization in discrete ray and field sampling



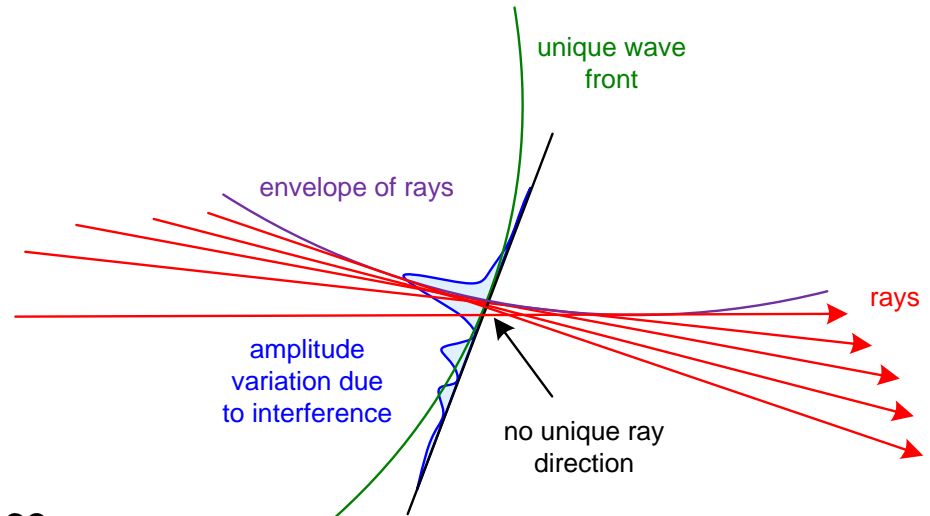
$$x_P = x_Q + W(x_Q) \tan \varphi$$

$$\Delta A_j = \Delta x \cdot \Delta y$$

$$\overline{PQ} = \frac{W(x_Q)}{\cos \varphi}$$

- Critical and limits of conversion in regions of a caustic,
Failure of conversion for crossing of rays

- Crossing rays
 - caustic surfaces
 - no unique ray direction
 - singular behavior of the eikonal with special solutions
- The singular solutions describes the envelope of the rays
- In the physical viewpoint, here interference takes place and the amplitude is no longer constant
- The caustic surface can be obtained from the eikonal equation by special techniques





Geometrical Optics Approximation

- Helmholtz wave equation

$$\Delta \vec{E}(\vec{r}) + k^2 \cdot \vec{E}(\vec{r}) = 0$$

- Split into phase and amplitude

$$E(\vec{r}) = A(\vec{r}) \cdot e^{-ik_o L(\vec{r})}$$

- The phase/optical path is given by

$$L = n \cdot \vec{s} \cdot \vec{r}$$

- Insertion, separation of real and imaginary part:

1. equation

$$2\nabla A \cdot \nabla L + A \cdot \nabla^2 L = 0$$

2. equation

$$k_o^2 \cdot \left(n^2(\vec{r}) - |\nabla L|^2 \right) \cdot A + \nabla^2 A = 0$$

- Approximation of geometrical optics

Gives the Eikonal equation for the description
of ray propagation
or with ray direction \vec{s}

$$a \gg \lambda, \quad \lambda \cdot \nabla_x A \ll 1, \quad \frac{1}{k_o} \cdot \nabla_x A \ll 1$$

$$n^2(\vec{r}) - |\nabla L|^2 = 0$$

$$n(\vec{r}) \cdot \vec{s} = \nabla L$$

- Violation of the geometrical optical approximation:

1. large values of $|\nabla A(\vec{r})|$ edges, diffraction takes place

2. large values of $|\nabla L(\vec{r})|$ focal points, source points with large angles

- Ansatz for Helmholtz wave equation with Eikonal L

$$E(\vec{r}) = E_o(\vec{r}) \cdot e^{-ik_o L(\vec{r})}$$

- Limiting case geometrical optic

$$\lambda_o \rightarrow 0$$

delivers the Eikonal equation:

Description of rays

Diffraction effects not included

$$(\nabla L)^2 = n^2(\vec{r})$$

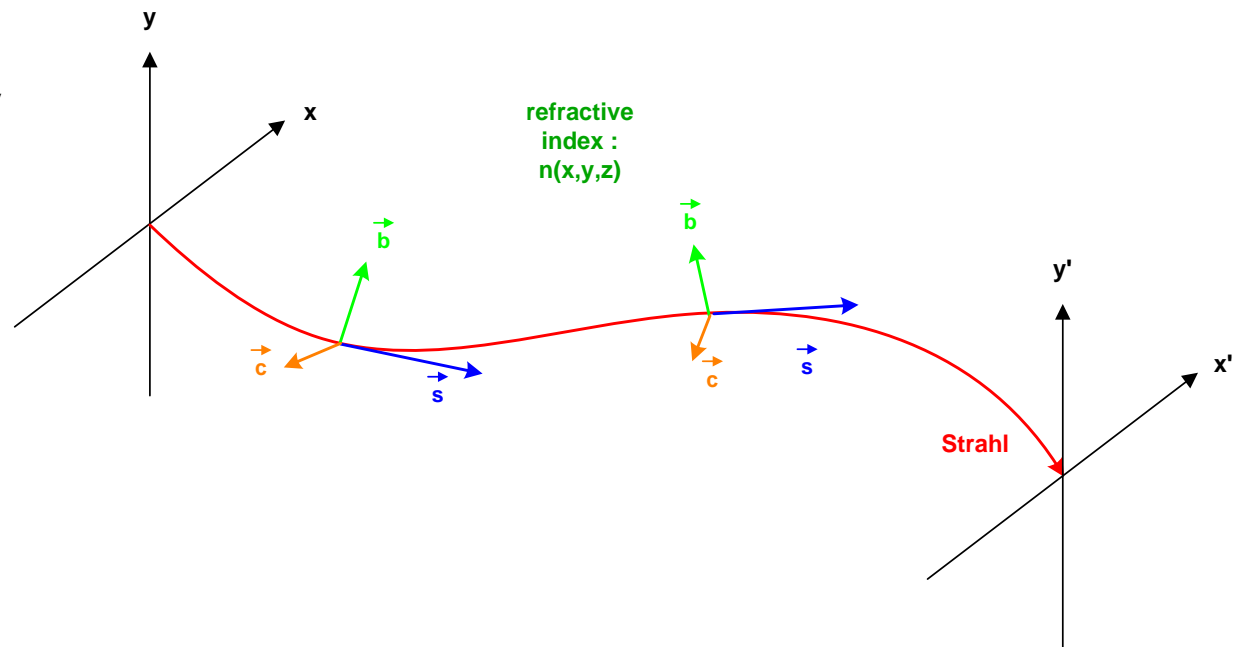
- L describes the optical path length
 $L = \text{const.}$ are the phase fronts of the wave

$$L = \int n(\vec{r}) d\vec{r}$$

- Application of the Eikonal equation:
Numerical solution for the raytracing in inhomogeneous media (gradient)
- Complex L : evanescent damped waves

- $$\frac{d^2 \vec{r}}{dt^2} = n \cdot \nabla n = \vec{D} = \begin{pmatrix} n \frac{\partial n}{\partial x} \\ n \frac{\partial n}{\partial y} \\ n \frac{\partial n}{\partial z} \end{pmatrix}$$

- Large computational times necessary for high accuracy





Raytracing in Grin Media

- Numerical solution of the eikonal equation in case of nonhomogeneous media

1. step width

$$\Delta t$$

2. scaled optical direction

$$\vec{T} = n \cdot \vec{s} = \begin{pmatrix} n \cdot s_x \\ n \cdot s_y \\ n \cdot s_z \end{pmatrix}$$

3. new position and direction

$$\vec{r}_{j+1} = \vec{r}_j + \Delta t \cdot \left(\vec{T}_j + \frac{\vec{A} + 2\vec{B}}{6} \right)$$

$$\vec{T}_{j+1} = \vec{T}_j + \frac{\vec{A} + 4\vec{B} + \vec{C}}{6}$$

4. Runge-Kutta parameters 4th order

$$\vec{A} = \Delta t \cdot \vec{D}(\vec{r}_j)$$

$$\vec{B} = \Delta t \cdot \vec{D} \left(\vec{r}_j + \frac{\Delta t \cdot \vec{T}_j}{2} + \frac{\Delta t \cdot \vec{A}}{8} \right)$$

$$\vec{C} = \Delta t \cdot \vec{D} \left(\vec{r}_j + \Delta t \cdot \vec{T}_j + \frac{\Delta t \cdot \vec{B}}{2} \right)$$

- Refocusing in parabolic profile
- Helical ray path in 3 dimensions

