



**Institute of
Applied Physics**

Friedrich-Schiller-Universität Jena

Imaging and Aberration Theory

Lecture 1: Paraxial optics

2018-10-19

Herbert Gross

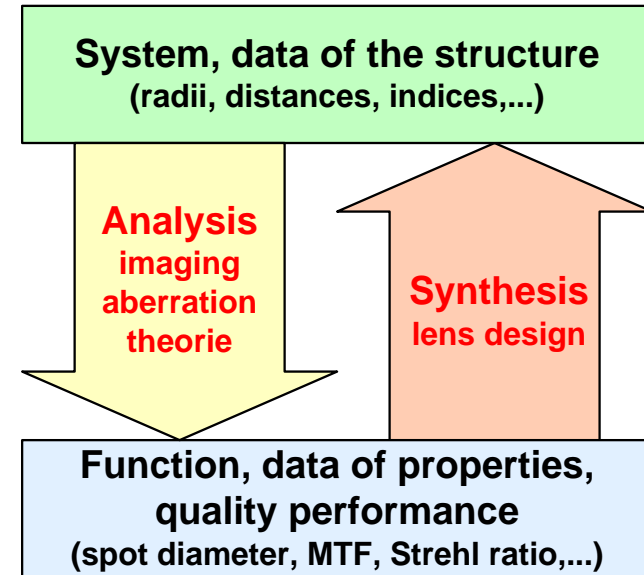
Schedule - Imaging and aberration theory 2018

1	19.10.	Paraxial imaging	paraxial optics, fundamental laws of geometrical imaging, compound systems
2	26.10.	Pupils, Fourier optics, Hamiltonian coordinates	pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates
3	02.11.	Eikonal	Fermat principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media
4	09.11.	Aberration expansions	single surface, general Taylor expansion, representations, various orders, stop shift formulas
5	16.11.	Representation of aberrations	different types of representations, fields of application, limitations and pitfalls, measurement of aberrations
6	23.11.	Spherical aberration	phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders
7	30.11.	Distortion and coma	phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options
8	07.12.	Astigmatism and curvature	phenomenology, Coddington equations, Petzval law, correction options
9	14.12.	Chromatical aberrations	Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum
10	21.12.	Sine condition, aplanatism and isoplanatism	Sine condition, isoplanatism, relation to coma and shift invariance, pupil aberrations, Herschel condition, relation to Fourier optics
11	11.01.	Wave aberrations	definition, various expansion forms, propagation of wave aberrations
12	18.01.	Zernike polynomials	special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, measurement
13	25.01.	Point spread function	ideal psf, psf with aberrations, Strehl ratio
14	01.02.	Transfer function	transfer function, resolution and contrast
15	08.02.	Additional topics	Vectorial aberrations, generalized surface contributions, Aldis theorem, intrinsic and induced aberrations, reverbability



1. Cardinal elements
2. Lens properties
3. Imaging, magnification
4. Afocal systems and telecentricity
5. Paraxial approximation
6. Matrix calculus

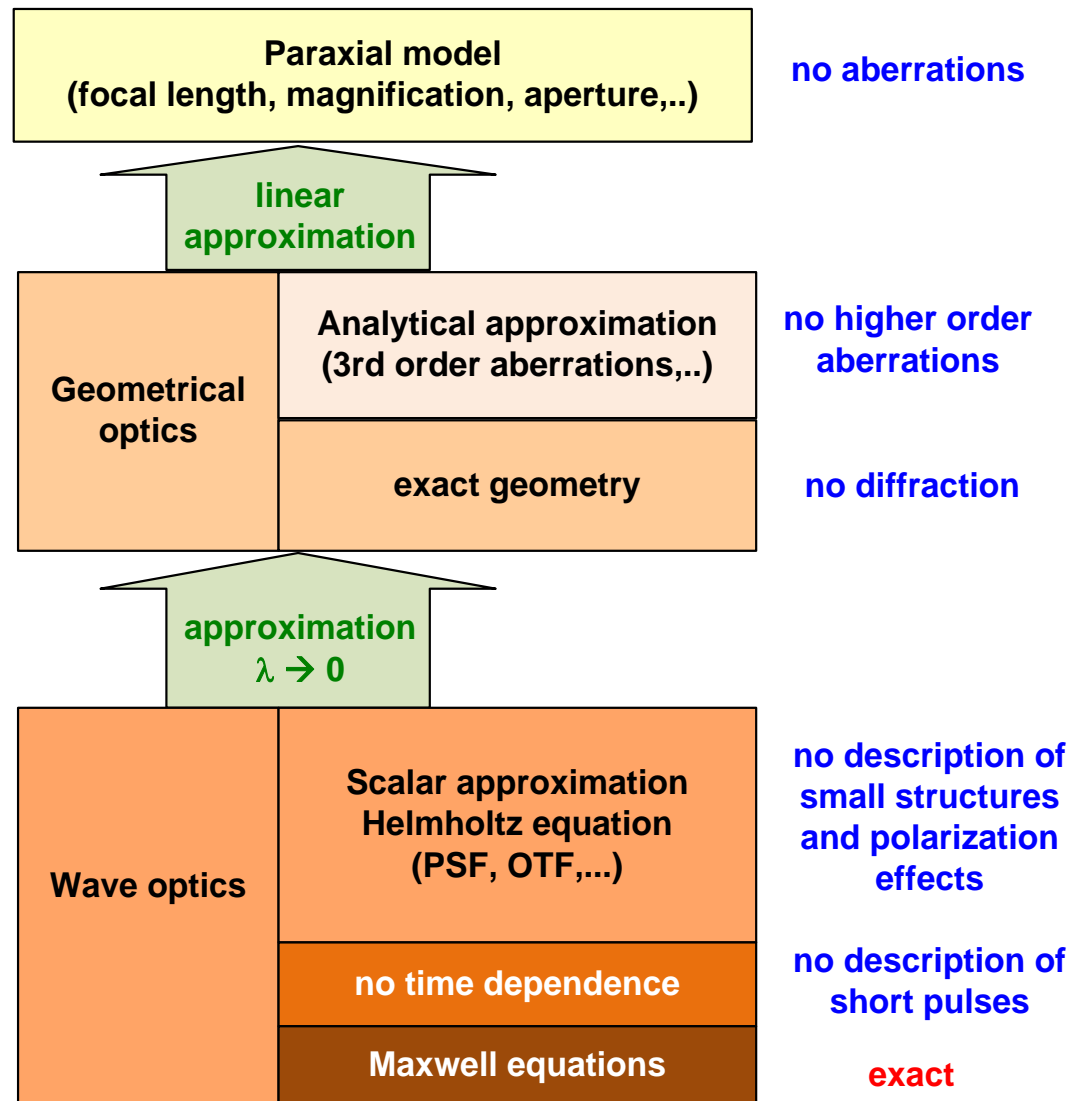
- Principal purpose of calculations:





Approximation of Optical Models

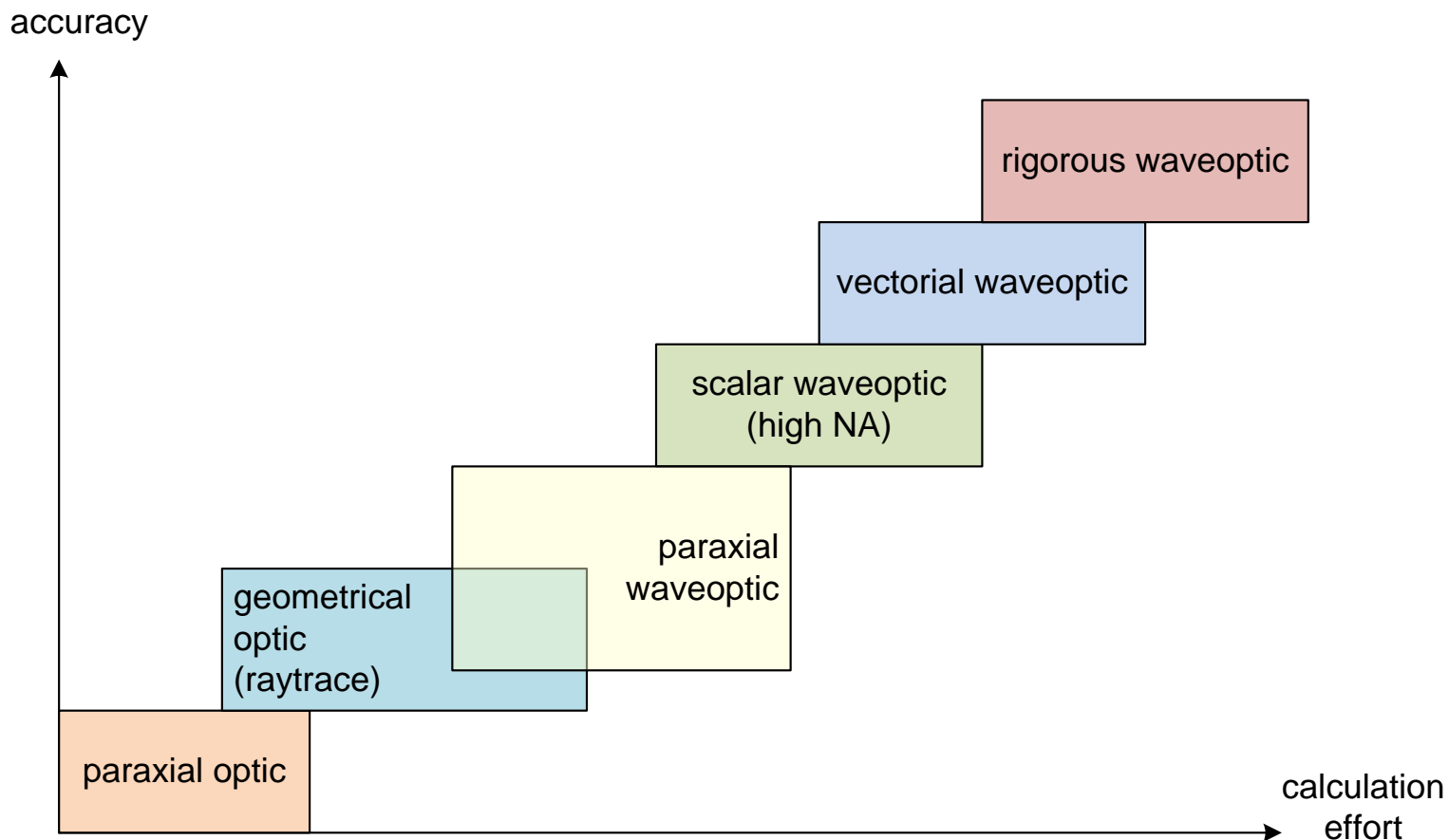
- Imaging model with levels of refinement





Model depth of Light Propagation

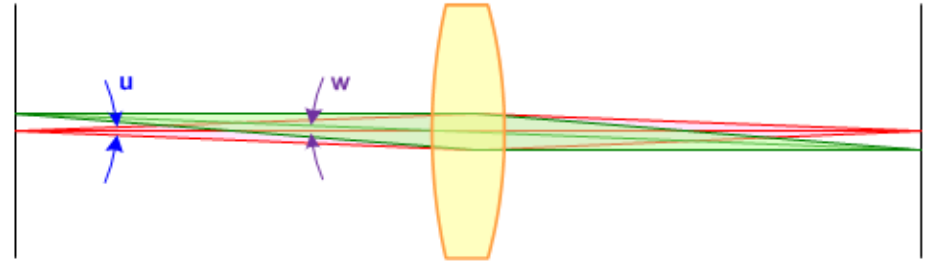
- Different levels of modelling in optical propagation
- Schematical illustration (not to scale)



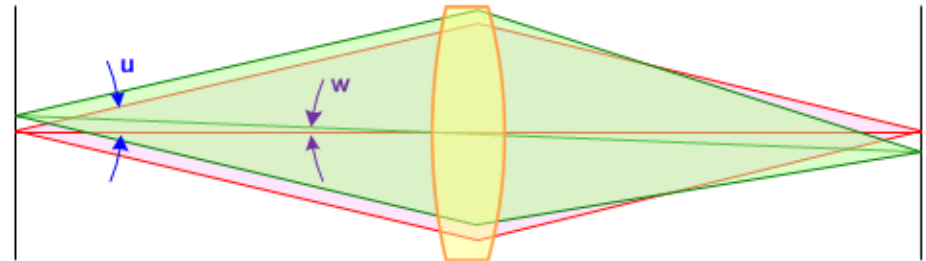
Paraxial Approximation

- Paraxiality is given for small angles relative to the optical axis for all rays
- Large numerical aperture angle u violates the paraxiality, spherical aberration occurs
- Large field angles w violates the paraxiality, coma, astigmatism, distortion, field curvature occurs

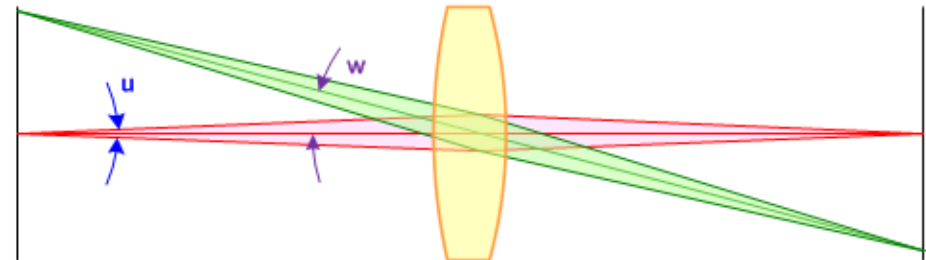
a) paraxial, small aperture / small field



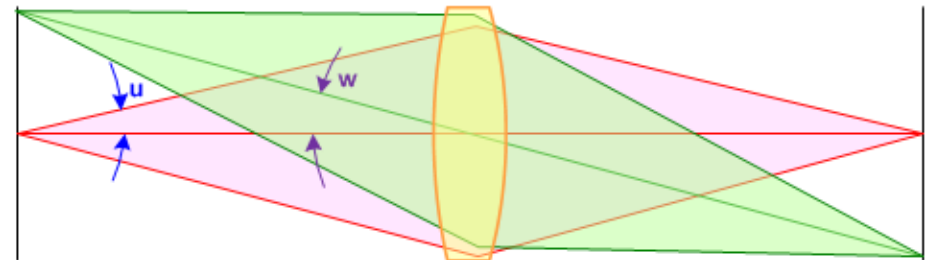
b) non-paraxial, large aperture / small field



c) non-paraxial, small aperture / large field



d) non-paraxial, large aperture / large field



Paraxial approximation:

- Law of refraction for finite angles I, I'

$$n \cdot \sin I = n' \cdot \sin I'$$

- sin-expansion

$$\sin i = i - \frac{i^3}{6} + \frac{i^5}{120} - \frac{i^7}{5040} + \frac{i^9}{362880} - \dots$$

- Small incidence angles allows for a linearization of the law of refraction
- Small angles of rays at every surface

$$n \cdot i = n' \cdot i'$$

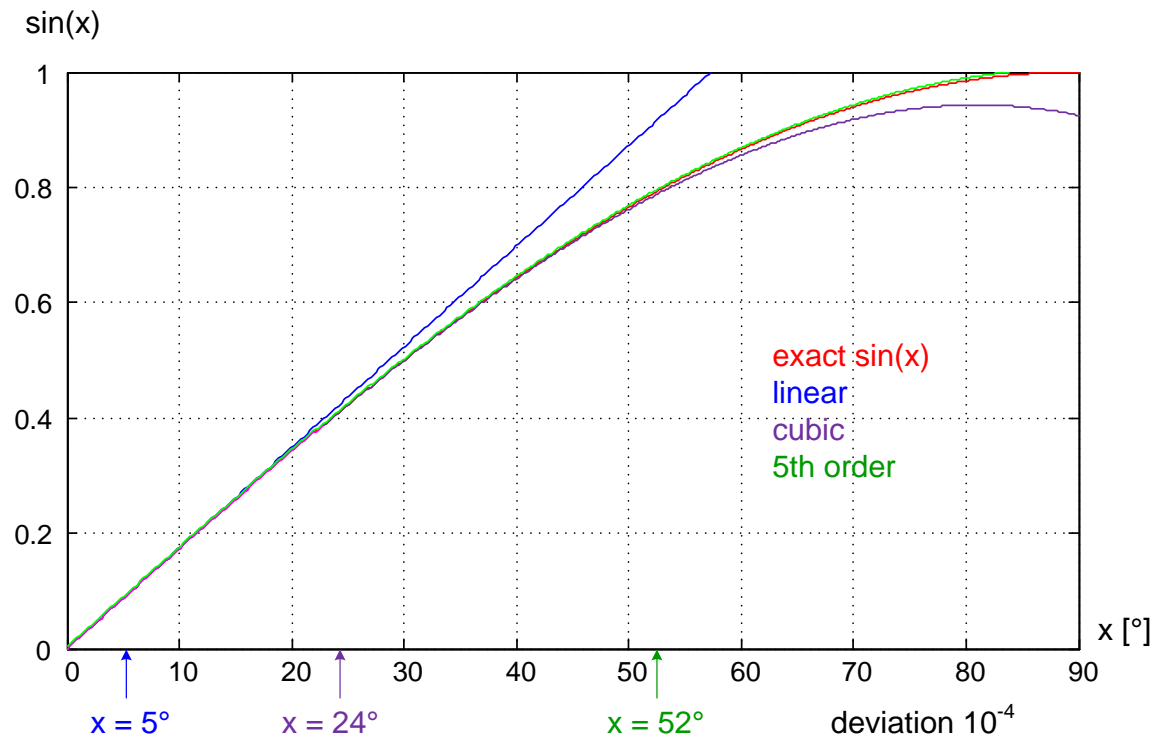
- All optical imaging conditions become linear (Gaussian optics), calculation with ABCD matrix calculus is possible
- No aberrations occur in optical systems
- There are no truncation effects due to transverse finite sized components
- Serves as a reference for ideal system conditions
- Is the fundament for many system properties (focal length, principal plane, magnification,...)
- The sag of optical surfaces (difference in z between vertex plane and real surface intersection point) can be neglected
- All waves are plane or spherical (parabolic)
- The phase factor of spherical waves is quadratic

$$E(x) = E_0 \cdot e^{-\frac{i\pi x^2}{\lambda R}}$$



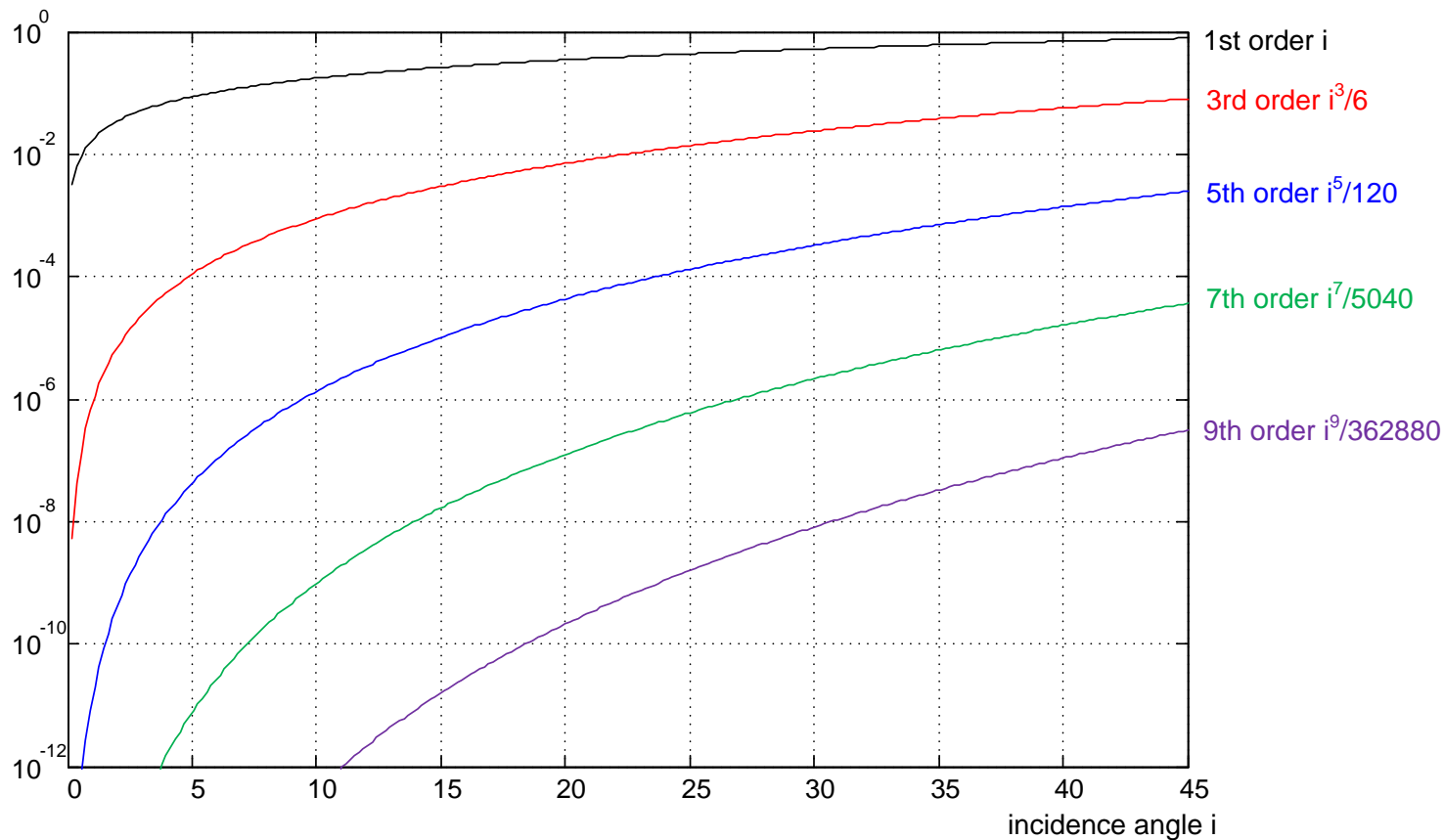
Paraxial approximation

- Taylor expansion of the sin-function
- Definition of allowed error 10^{-4}
- Deviation of the various approximations:
 - linear: 5°
 - cubic: 24°
 - 5th order: 542°



- Contribution of various orders of the sin-expansion

$$\sin i = i - \frac{i^3}{6} + \frac{i^5}{120} - \frac{i^7}{5040} + \frac{i^9}{362880} - \dots$$





Paraxial approximation

- Law of refraction for finite angles I , I'

$$n \cdot \sin I = n' \cdot \sin I'$$

- Taylor expansion

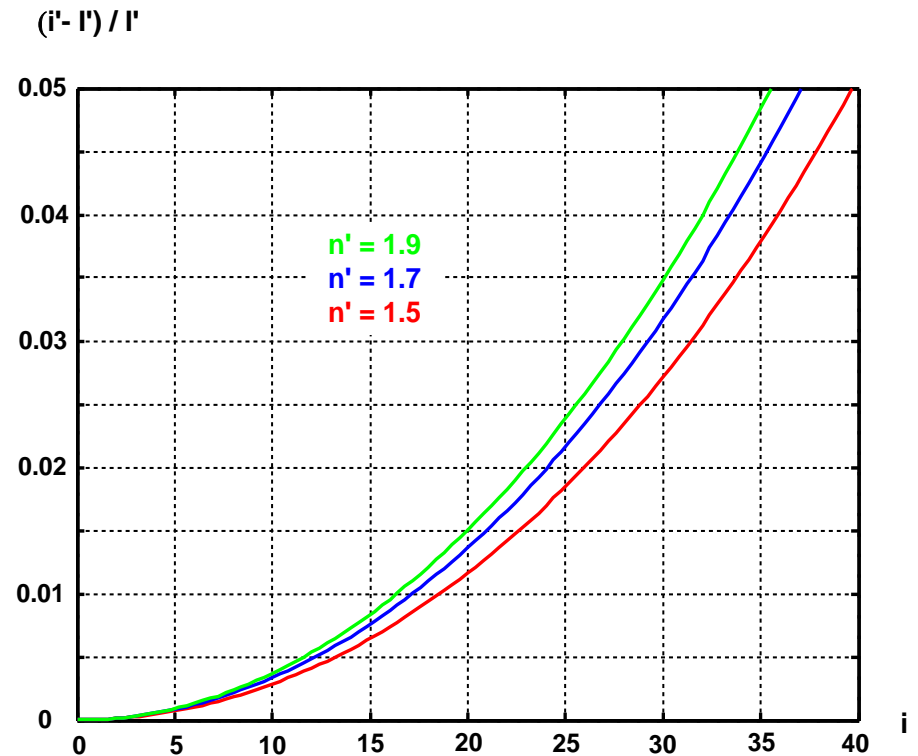
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

- Linear formulation of the law of refraction for small angles i , i'

$$n \cdot i = n' \cdot i'$$

- Relative direction error of the paraxial approximation

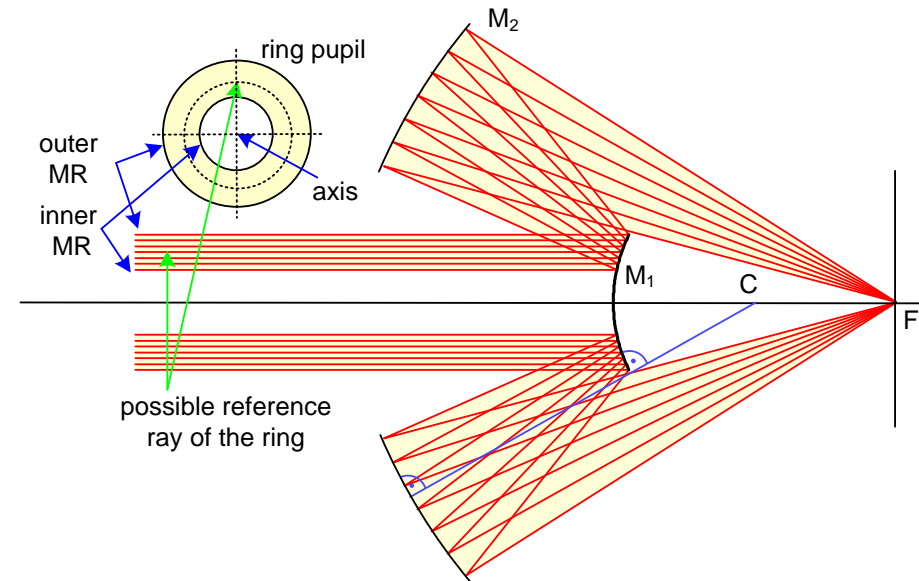
$$\varepsilon = \frac{i' - I'}{I'} = \frac{\frac{n \cdot i}{n'}}{\arcsin\left(\frac{n \cdot \sin i}{n'}\right)} - 1$$



- Pitfalls in the classical definition of paraxiality:

1. Central obscuration and ring-shaped pupil:

- Paraxial marginal ray of no relevance
- Reference on centroid ray



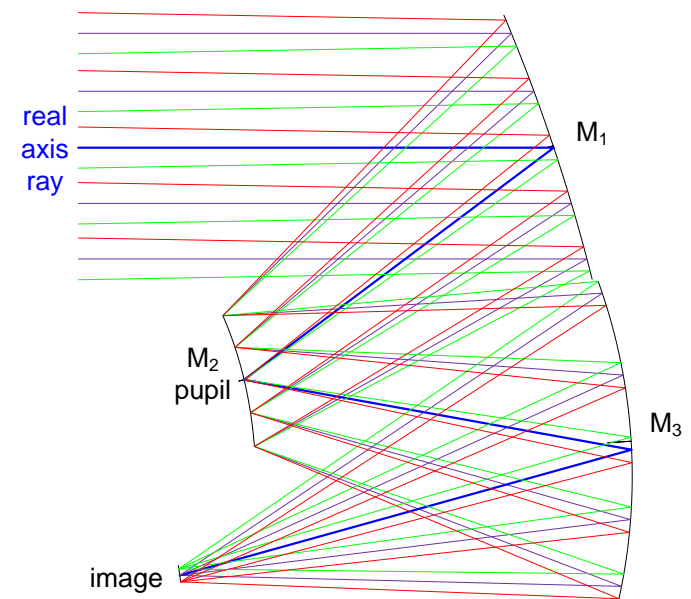
2. General 3D system without straight axis:

Central ray as reference, calculated finite paraxial rays in the neighborhood of the real chief ray

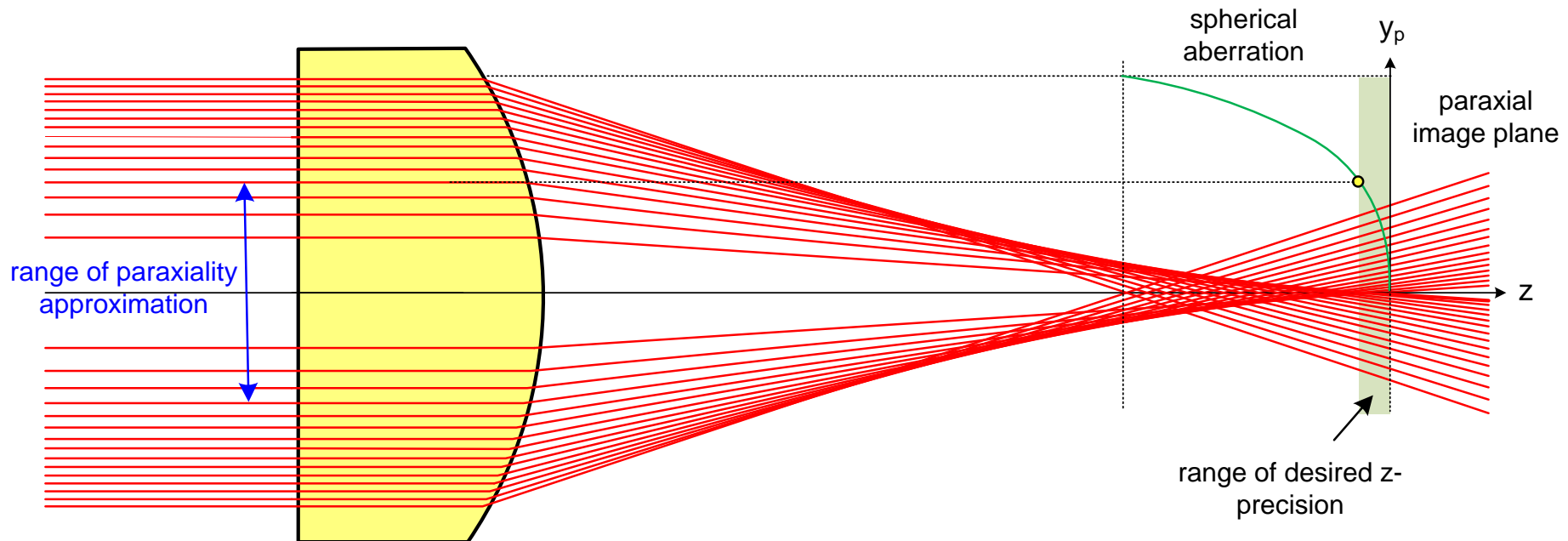
Distortion information is lost

- General quasi-parabasal rays:

- macroscopic astigmatism
- aberrations reference definition more complicated
- separated view on chief ray / marginal ray



- Illustration of validity range of paraxial approximation
- Desired accuracy in z defines lateral range of corresponding caustic
- All rays inside this range are fine with the needed precision and can be calculated as paraxial



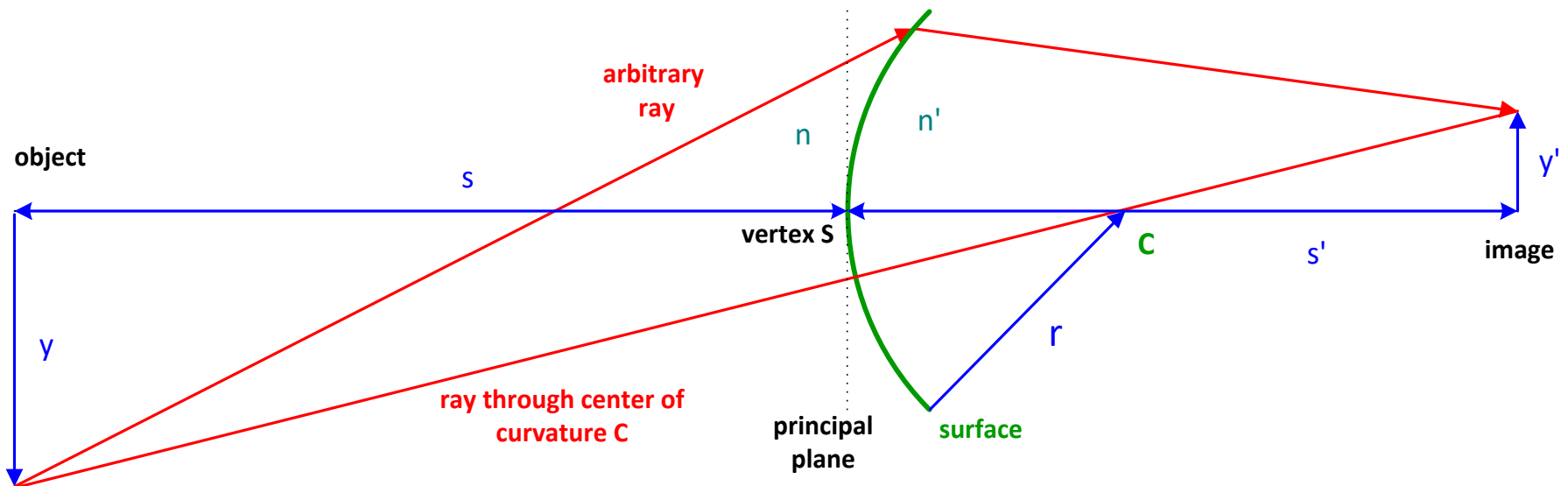


Single Surface

- Single surface between two media
Radius r , refractive indices n , n'
- Imaging condition, paraxial
- Abbe invariant
alternative representation of the
imaging equation

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{r} = \frac{1}{f'}$$

$$Q_s = n \cdot \left(\frac{1}{r} - \frac{1}{s} \right) = n' \cdot \left(\frac{1}{r} - \frac{1}{s'} \right)$$





Notations of a lens

P principal point

S vertex of the surface

F focal point

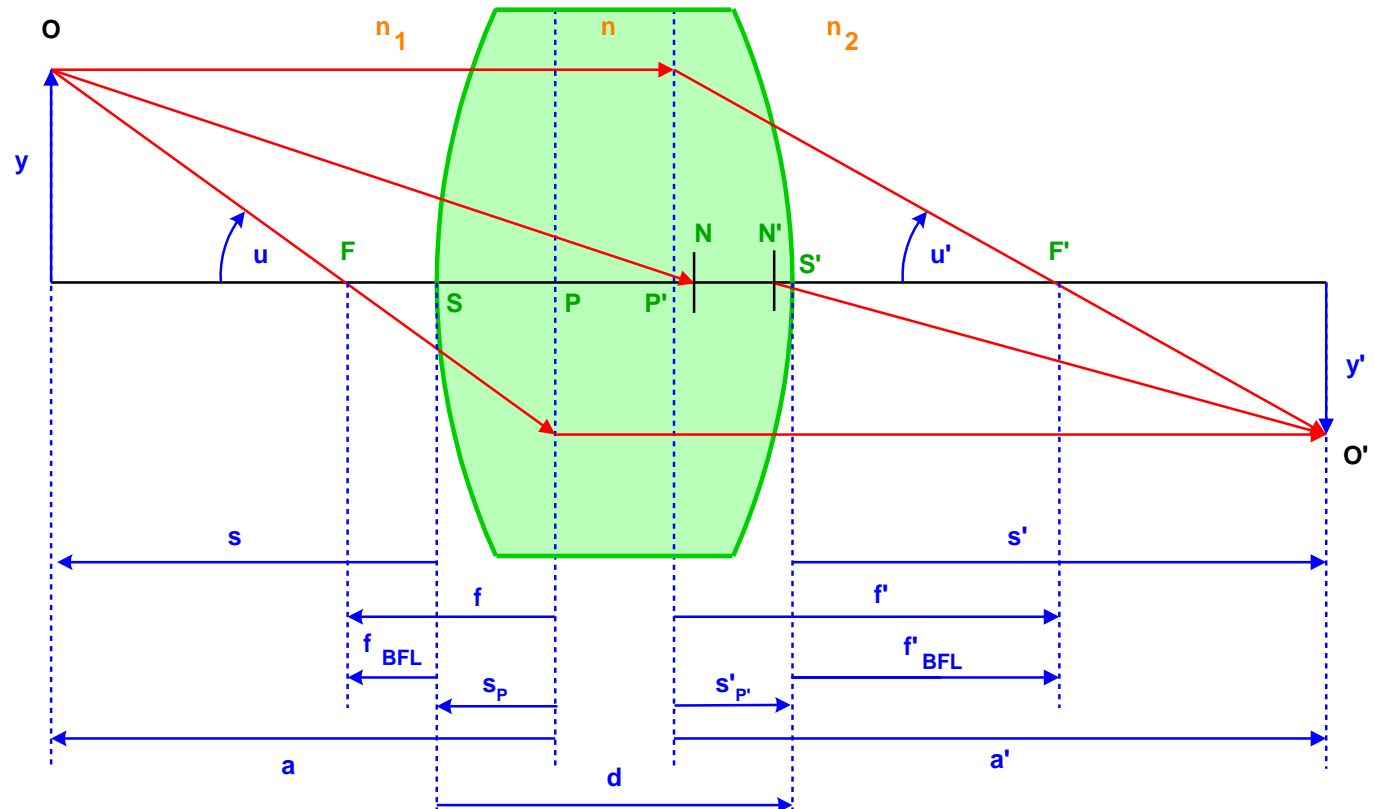
s intersection point
of a ray with axis

f focal length PF

r radius of surface
curvature

d thickness SS'

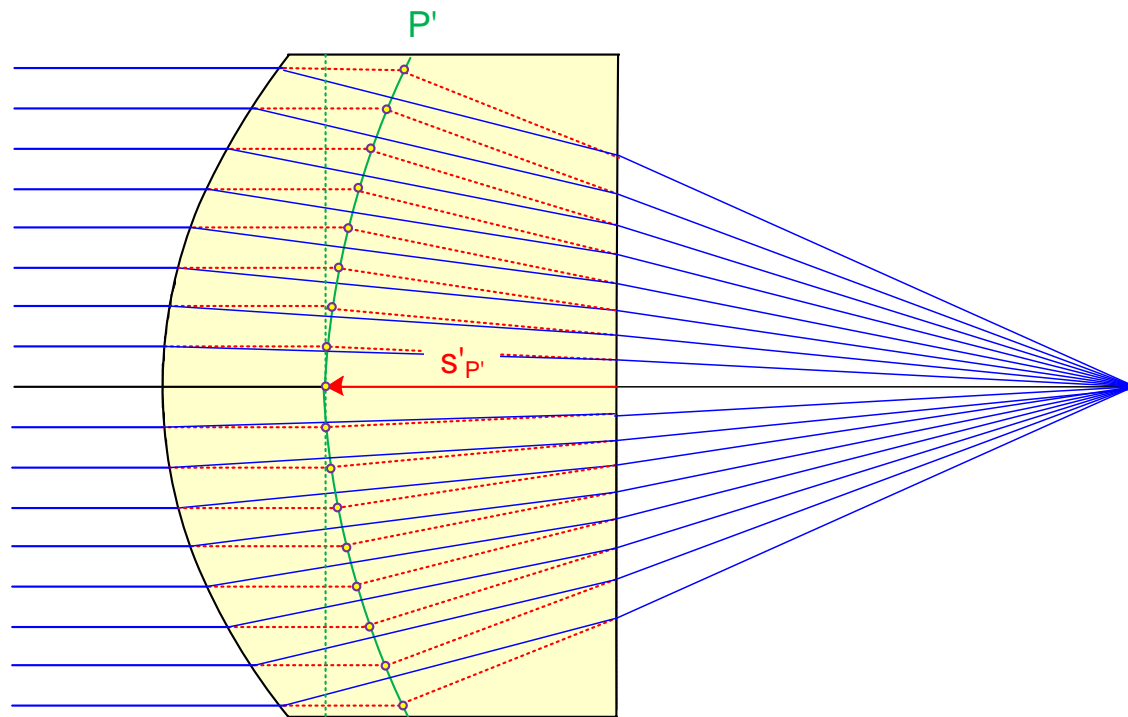
n refractive index



Cardinal Points of a Lens

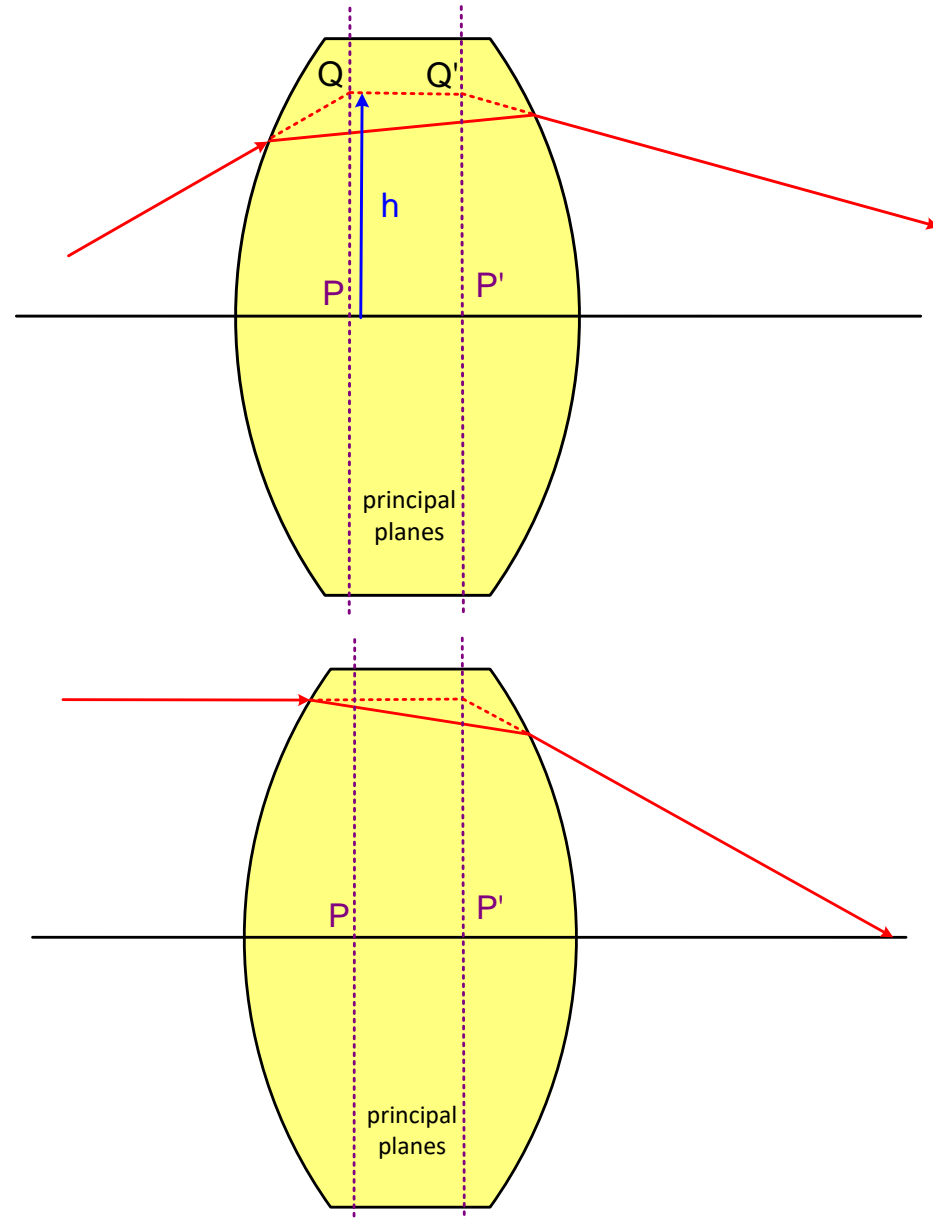


- Real lenses:
The surface with the principal points as apparent ray bending points is a curved shell
- The ideal principal plane exists only in the paraxial approximation

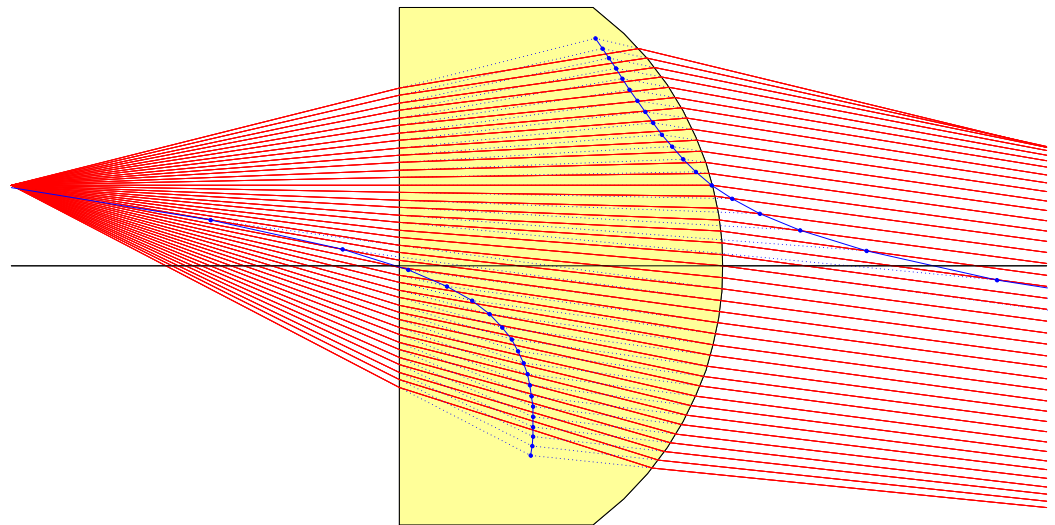


Cardinal Elements of a Lens

- Principal plane P:
incoming ray hits intersection point with
P is transferred with the same height h
to P'
- Special case of incident ray parallel
to the axis:
principal plane P':
location of apparent ray bending



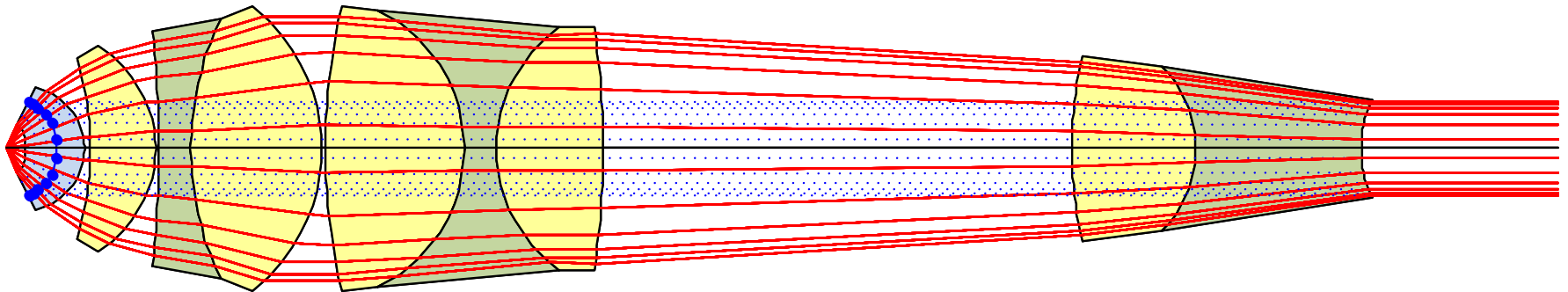
- The principal planes in paraxial optics are defined as the locations of the apparent ray bending of a lens or system
- In the case of a system with corrected sine conditions, these surfaces are spheres
- Sine condition and pupil spheres are also limited for off-axis points near to the optical axis
- For object points far from the axis, the apparent locations are complicated surfaces, which may consist of two branches





Limitation of Principal Surface Definition

- A real microscopic lens has a nearly perfect spherical shape of the principal surface





Main properties of a lens

- Main notations and properties of a lens:

- radii of curvature r_1 , r_2
curvatures c
sign: $r > 0$: center of curvature
is located on the right side
- thickness d along the axis
- diameter D
- index of refraction of lens material n

$$c_1 = \frac{1}{r_1} \quad c_2 = \frac{1}{r_2}$$

- Focal length (paraxial)

$$f = \frac{y_F'}{\tan u} \quad , \quad f' = \frac{y}{\tan u'}$$

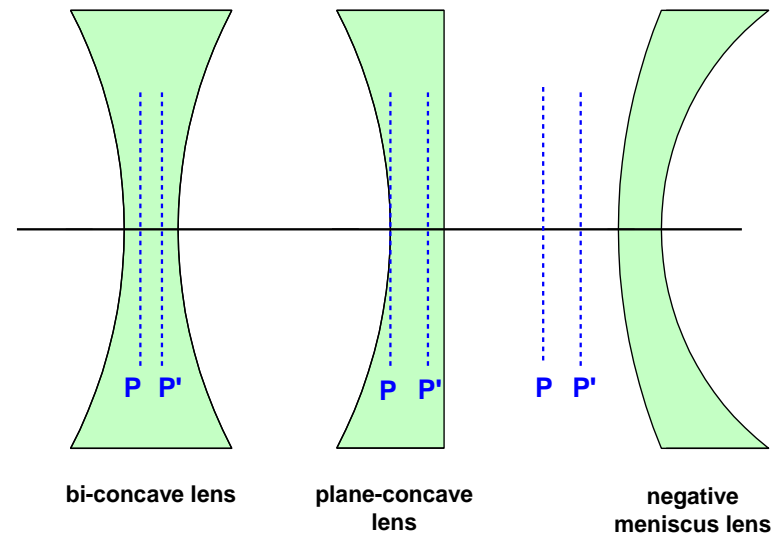
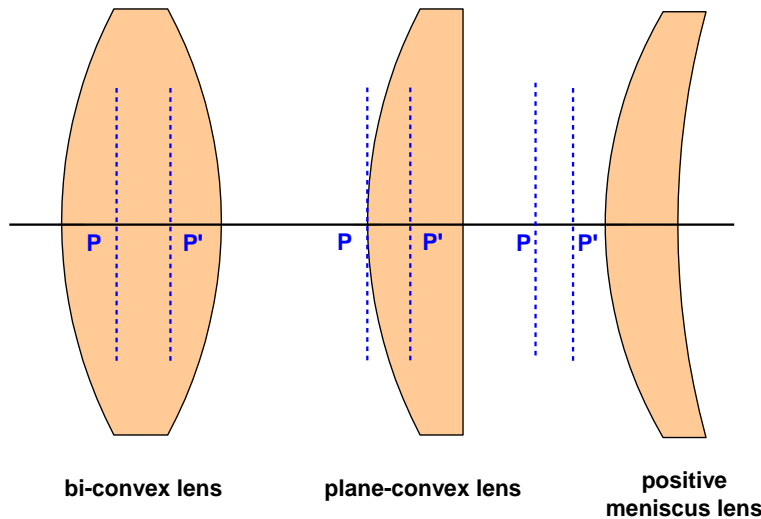
- Optical power

$$F = -\frac{n}{f} = \frac{n'}{f'}$$

- Back focal length
intersection length,
measured from the vertex point

$$s_{F'} = f' + s_{P'}$$

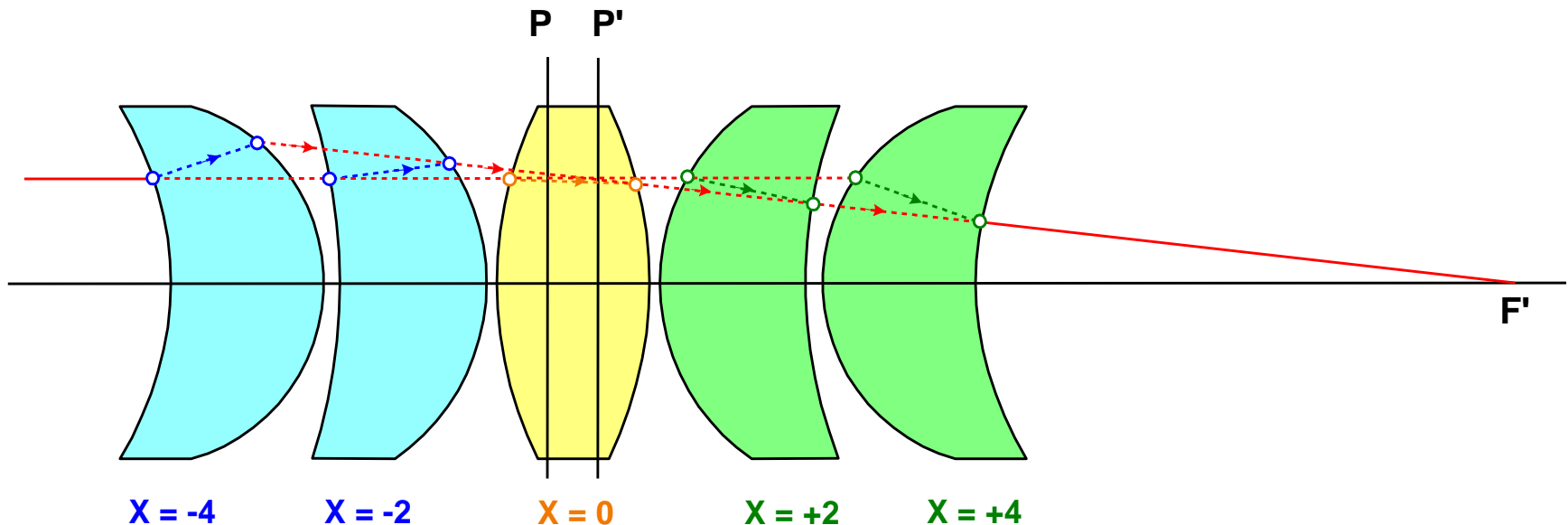
- Different shapes of singlet lenses:
 1. bi-, symmetric
 2. plane convex / concave, one surface plane
 3. Meniscus, both surface radii with the same sign
- Convex: bending outside
Concave: hollow surface
- Principal planes P , P' : outside for meniscus shaped lenses





Lens bending und shift of principal plane

- Ray path at a lens of constant focal length and different bending
- The ray angle inside the lens changes
- The ray incidence angles at the surfaces changes strongly
- The principal planes move
For invariant location of P, P' the position of the lens moves





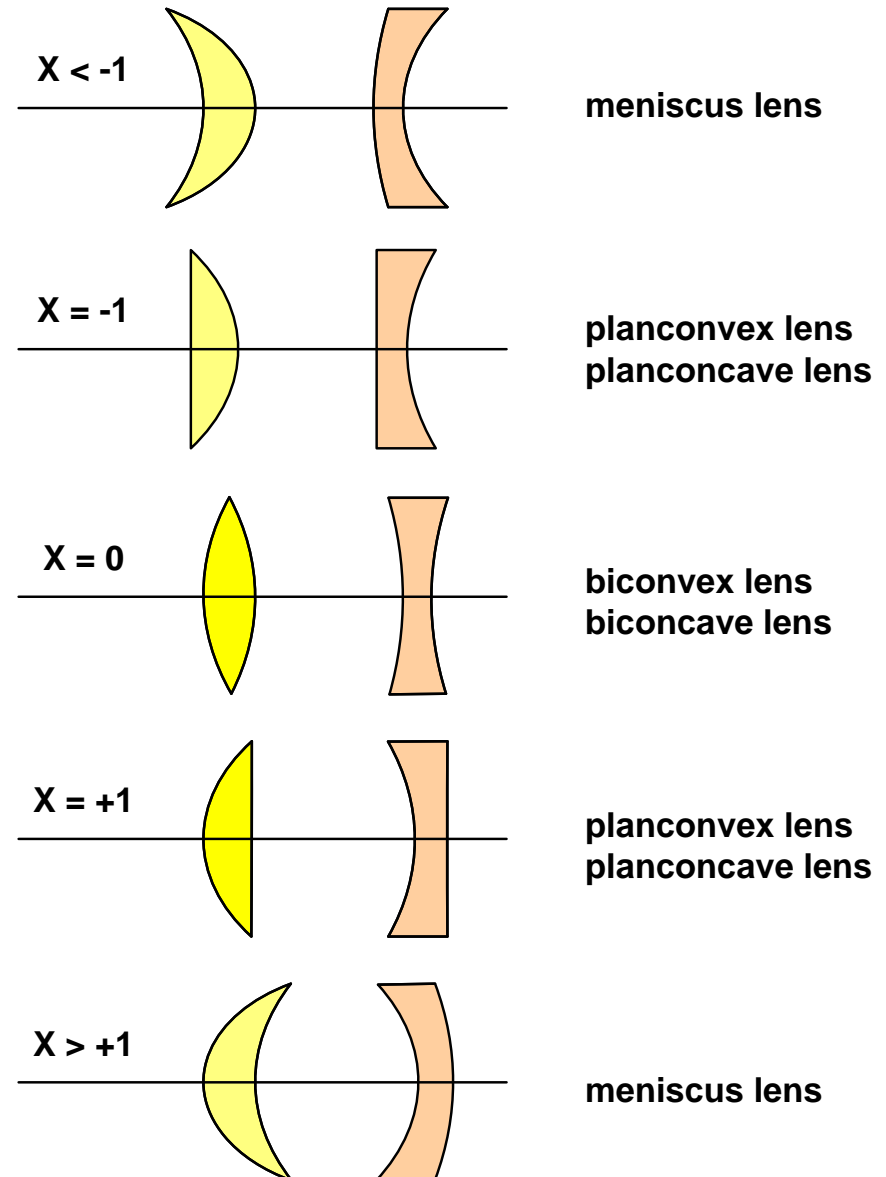
Bending of a Lens

- Bending: change of shape for invariant focal length

- Parameter of bending

$$X = \frac{R_1 + R_2}{R_2 - R_1}$$

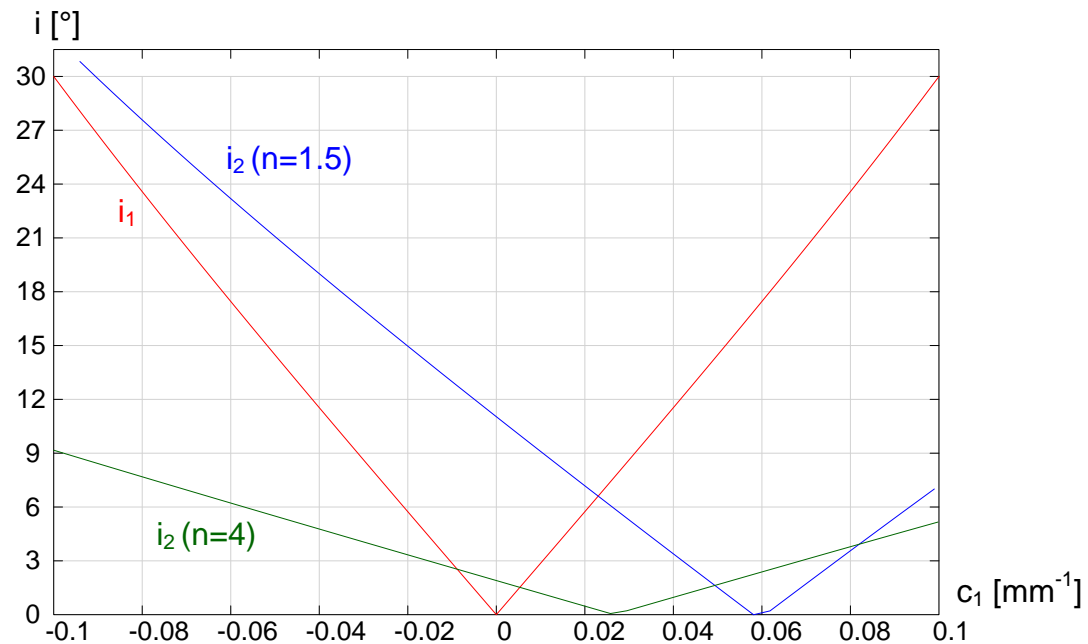
- Principal planes are moving
- Incidence angles and most aberrations are changing





Incidence of Bended Lens

- Changes of the incidence angles at the front and the rear surface of a bended lens
- Figure without sign of incidence angle
- Angle at the second surface depends on the refractive index

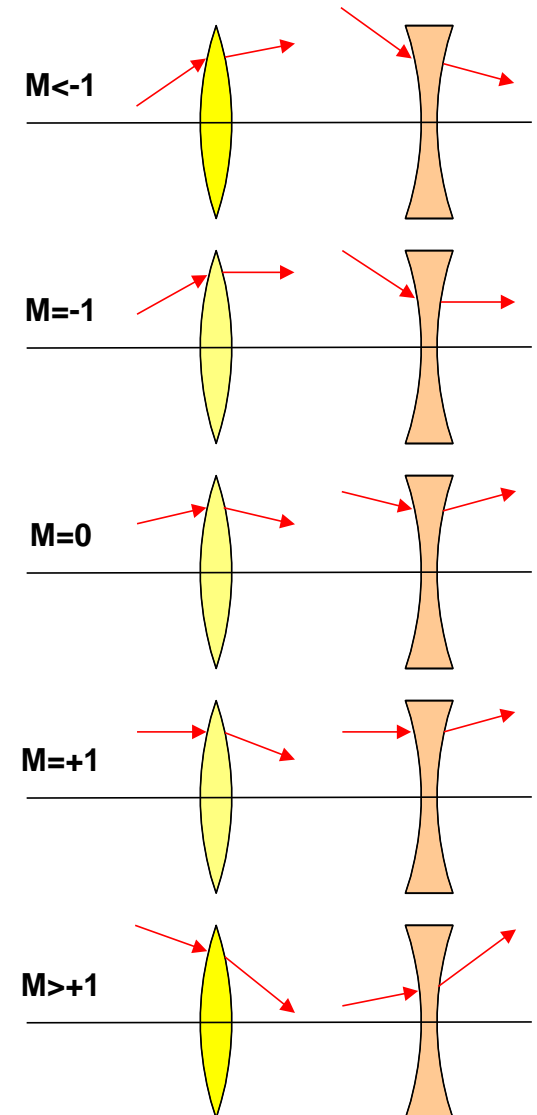


Magnification Parameter

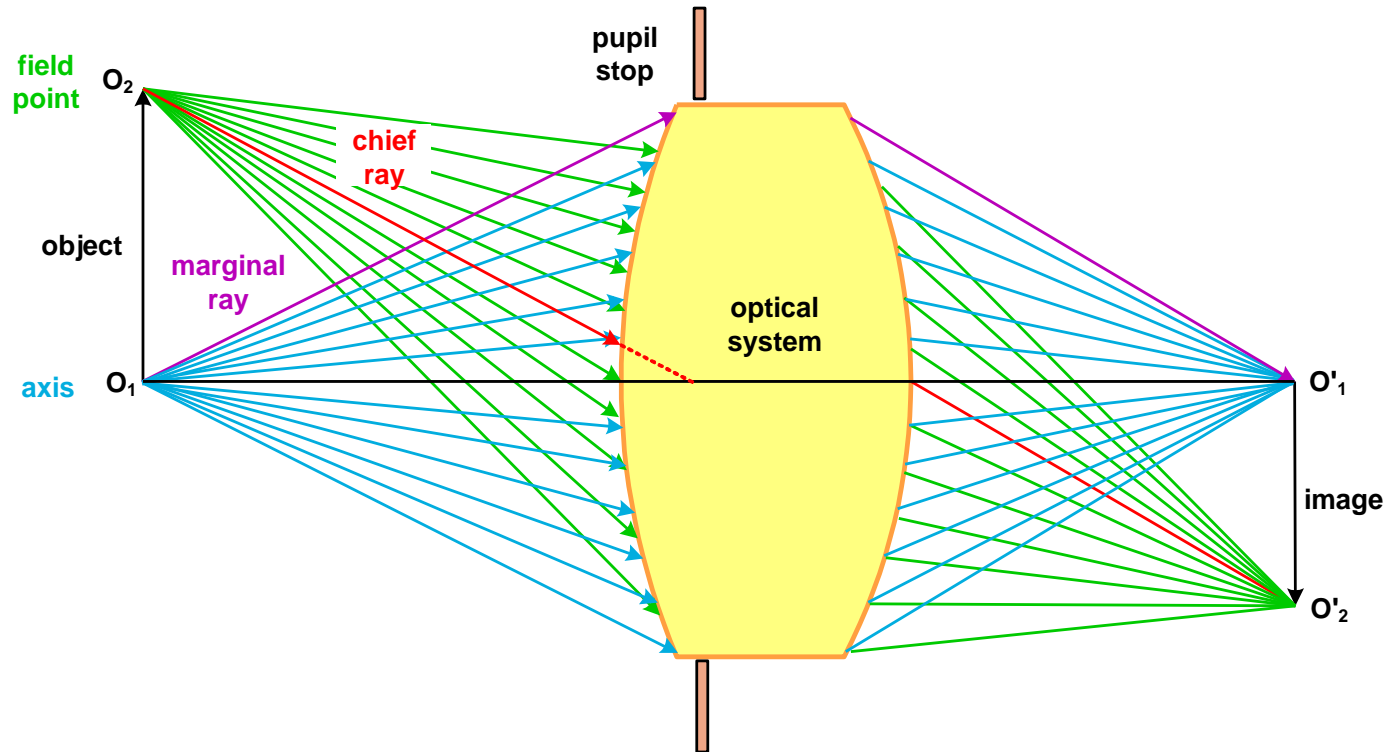
- Magnification parameter M :
defines ray path through the lens

$$M = \frac{U'+U}{U'-U} = \frac{1+m}{1-m} = \frac{2f}{s} + 1 = \frac{2f}{s'} - 1$$

- Special cases:
 1. $M = 0$: symmetrical 4f-imaging setup
 2. $M = -1$: object in front focal plane
 3. $M = +1$: object in infinity
- The parameter M strongly influences the aberrations



- Optical Image formation:
All ray emerging from one object point meet in the perfect image point
- Region near axis:
gaussian imaging
ideal, paraxial
- Image field size:
Chief ray
- Aperture/size of
light cone:
marginal ray
defined by pupil
stop



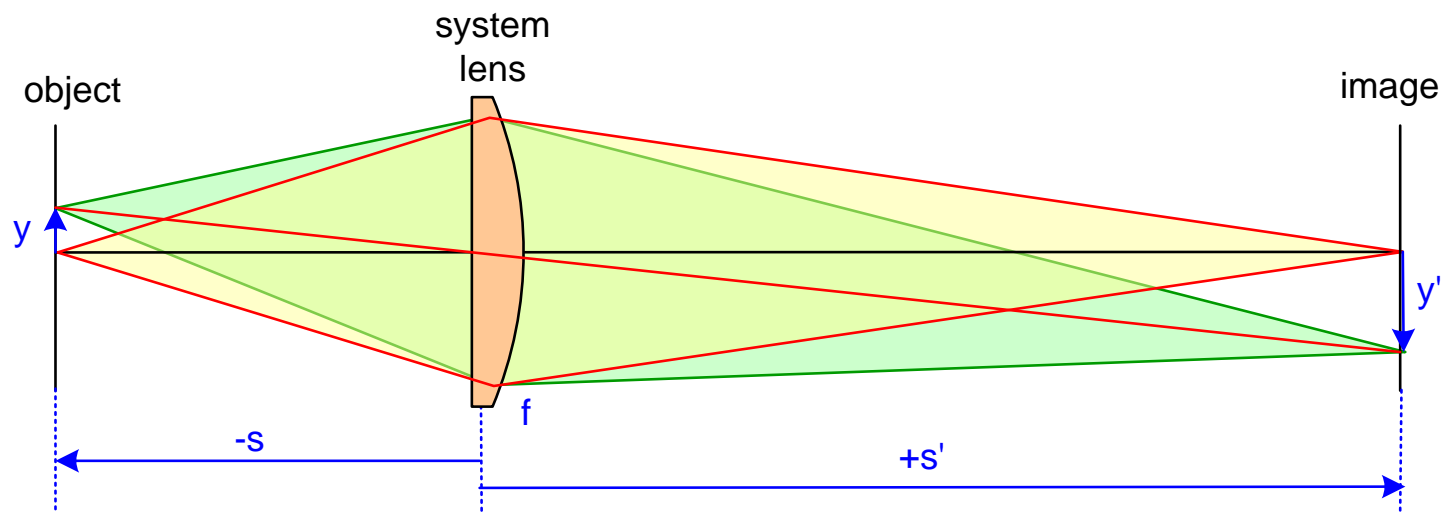
- Imaging with a lens

- Location of the image:
lens equation

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}$$

- Size of the image:
Magnification

$$m = \frac{y'}{y} = \frac{s'}{s}$$





Formulas for surface and lens imaging

- Single surface imaging equation

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{r} = \frac{1}{f'}$$

- Thin lens in air focal length

$$\frac{1}{f'} = (n - 1) \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

- Thin lens in air with one plane surface, focal length

$$f' = \frac{r}{n - 1}$$

- Thin symmetrical bi-lens

$$f' = \frac{r}{2 \cdot (n - 1)}$$

- Thick lens in air focal length

$$\frac{1}{f'} = (n - 1) \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{(n - 1)^2 d}{n \cdot r_1 r_2}$$



Imaging equation

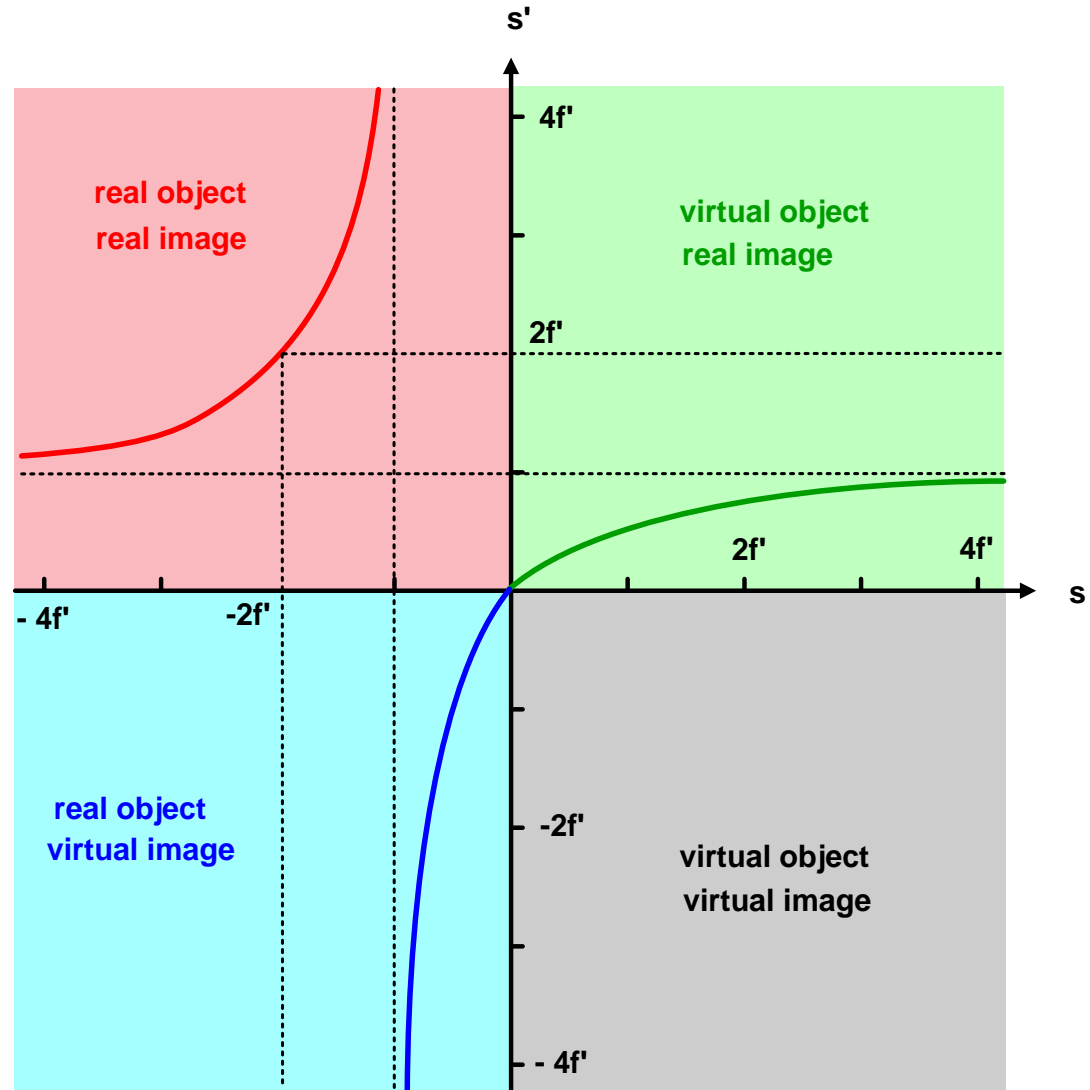
- Imaging by a lens in air:
lens makers formula

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}$$

- Magnification

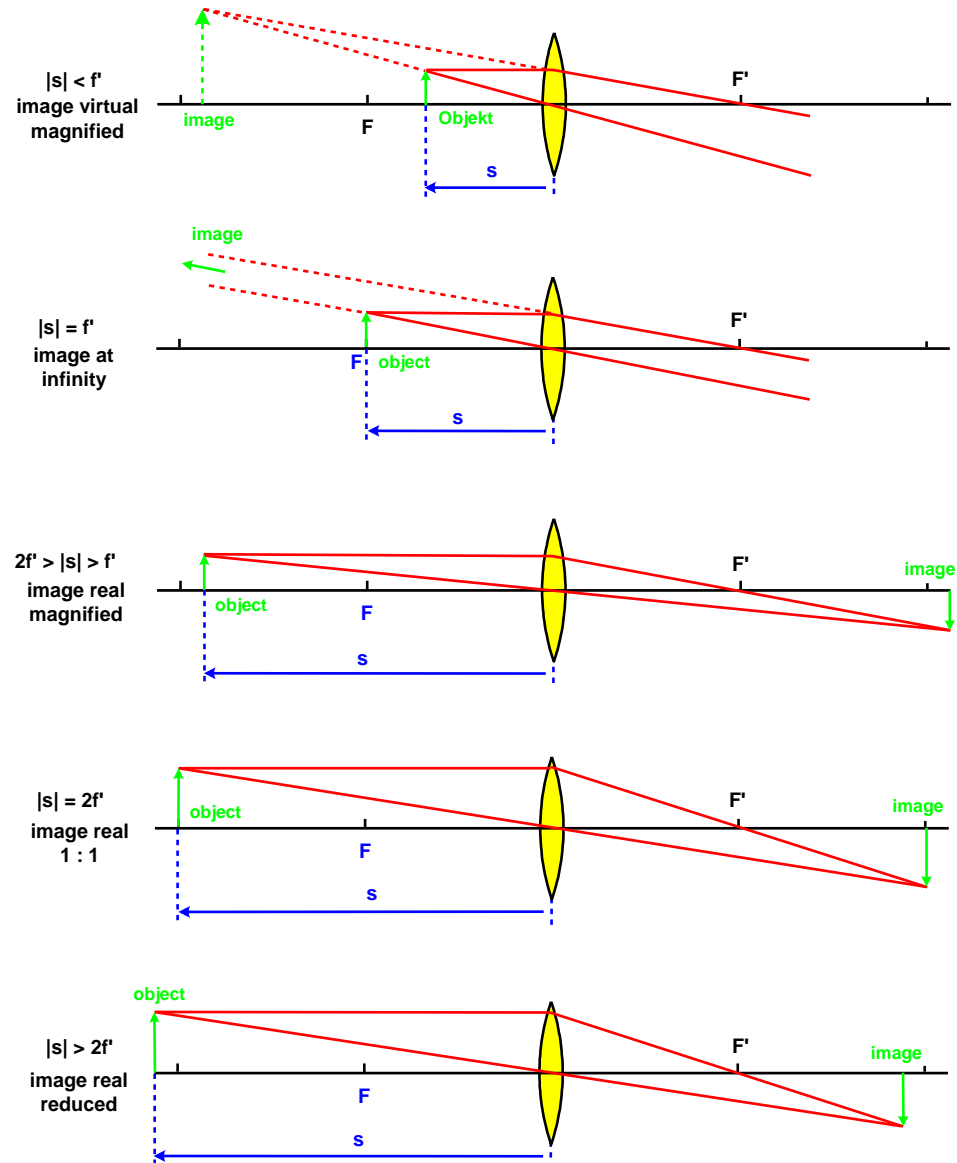
$$m = \frac{s'}{s}$$

- Real imaging:
 $s < 0$, $s' > 0$
- Intersection lengths s , s'
measured with respect to the
principal planes P , P'



Imaging by a Lens

- Ranges of imaging
Location of the image for a single lens system
- Change of object location
- Image could be:
 1. real / virtual
 2. enlarged/reduced
 3. in finite/infinite distance



- Two lenses with distance d

$$F = F_1 + F_2 - \frac{d \cdot F_1 \cdot F_2}{n}$$

- Focal length
distance of inner focal points e

$$f = \frac{f_1 \cdot f_2}{f_1 + f_2 - d} = \frac{f_1 \cdot f_2}{e}$$

- Sequence of thin lenses close together

$$F = \sum_k F_k$$

- Sequence of surfaces with relative ray heights h_j , paraxial

$$F = \sum_k \frac{h_k}{h_1} \cdot (n'_k - n_k) \cdot \frac{1}{r_k}$$

- Magnification

$$m = \frac{s'_1}{s_1} \cdot \frac{s'_2}{s_2} \dots \frac{s'_k}{s_k} \cdot \frac{n_1}{n'_k}$$



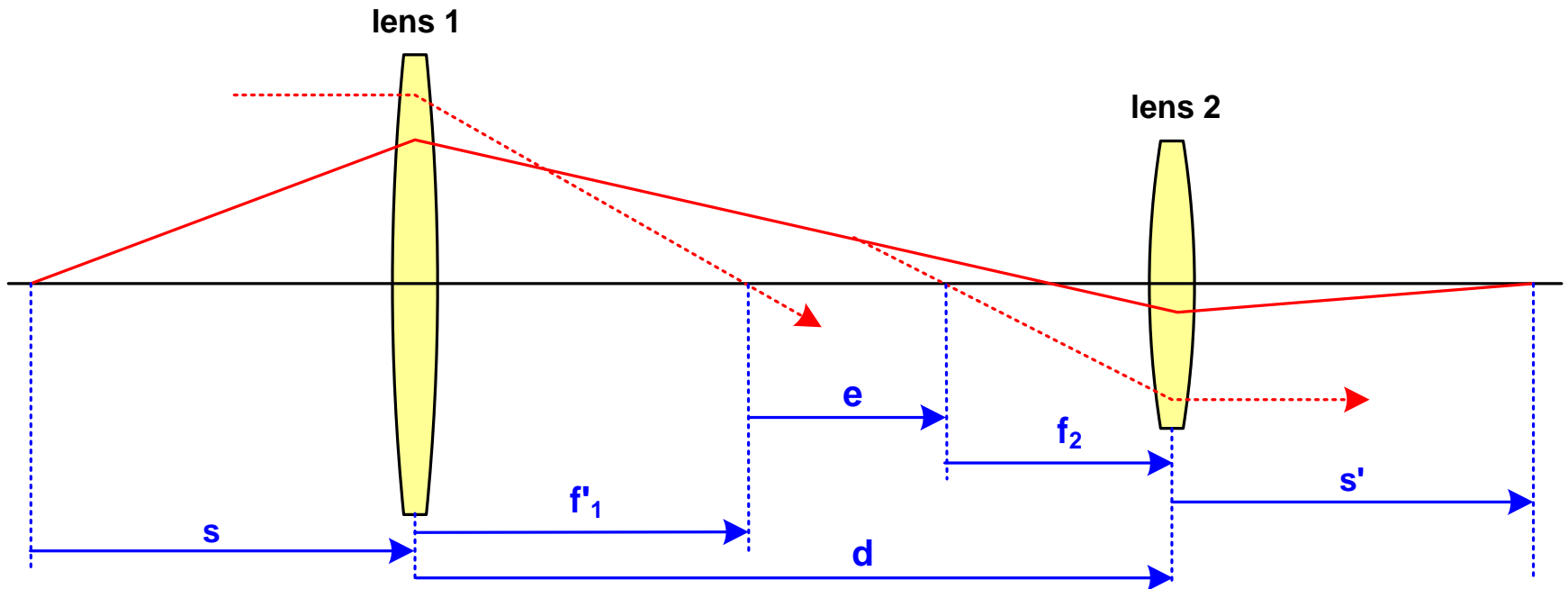
Two-Lens System

- Focal length
e: tube length

$$f' = \frac{f'_1 \cdot f'_2}{f'_1 + f'_2 - d} = \frac{f'_1 \cdot f'_2}{e}$$

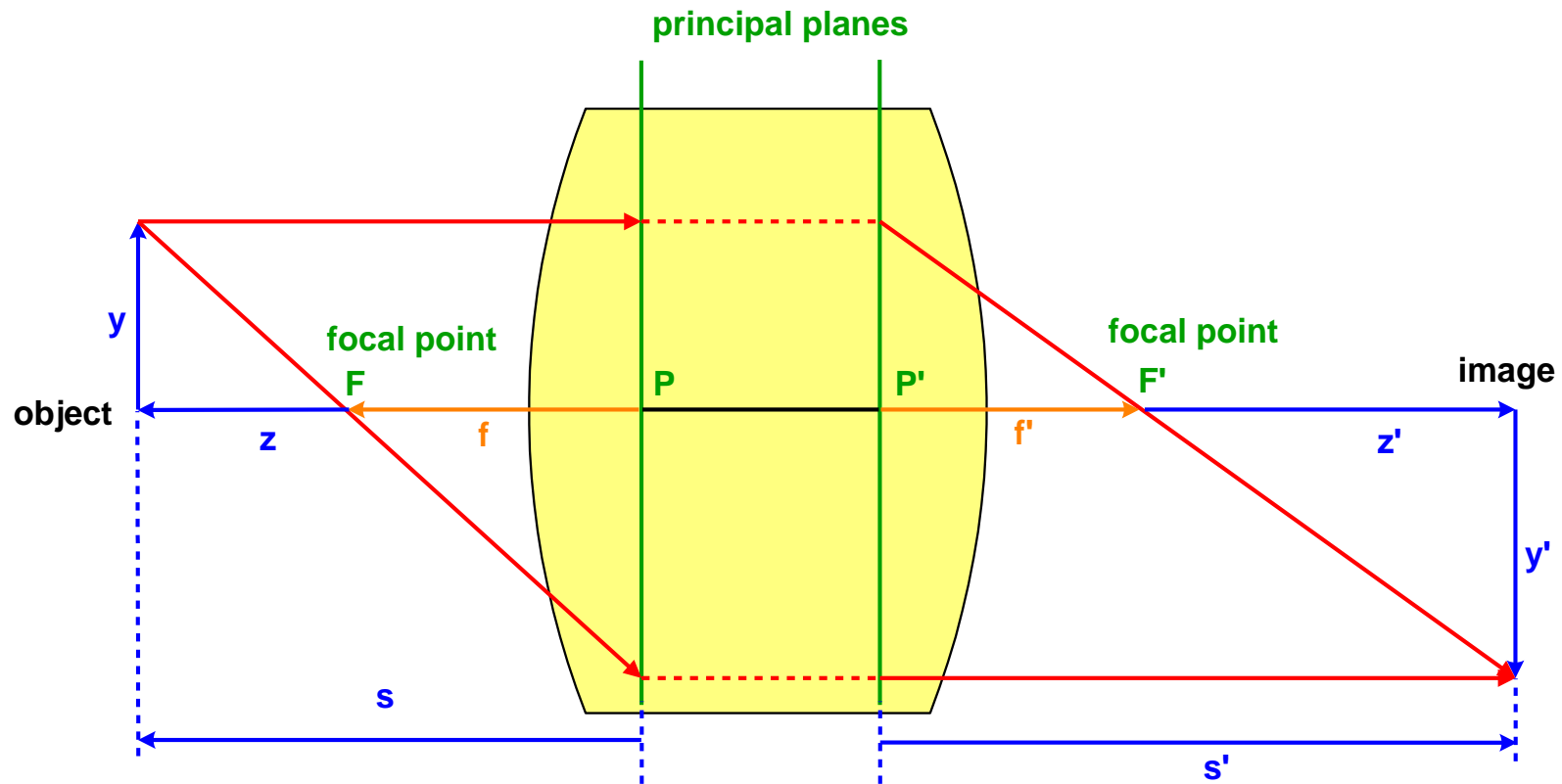
- Image location

$$s'_2 = \frac{(f'_1 - d) \cdot f'_2}{f'_1 + f'_2 - d} = \frac{(f'_1 - d) \cdot f'_2}{f'_1}$$



- Lateral magnification for finite imaging
- Scaling of image size

$$m = \frac{y'}{y} = - \frac{f \cdot \tan u}{f' \cdot \tan u'}$$

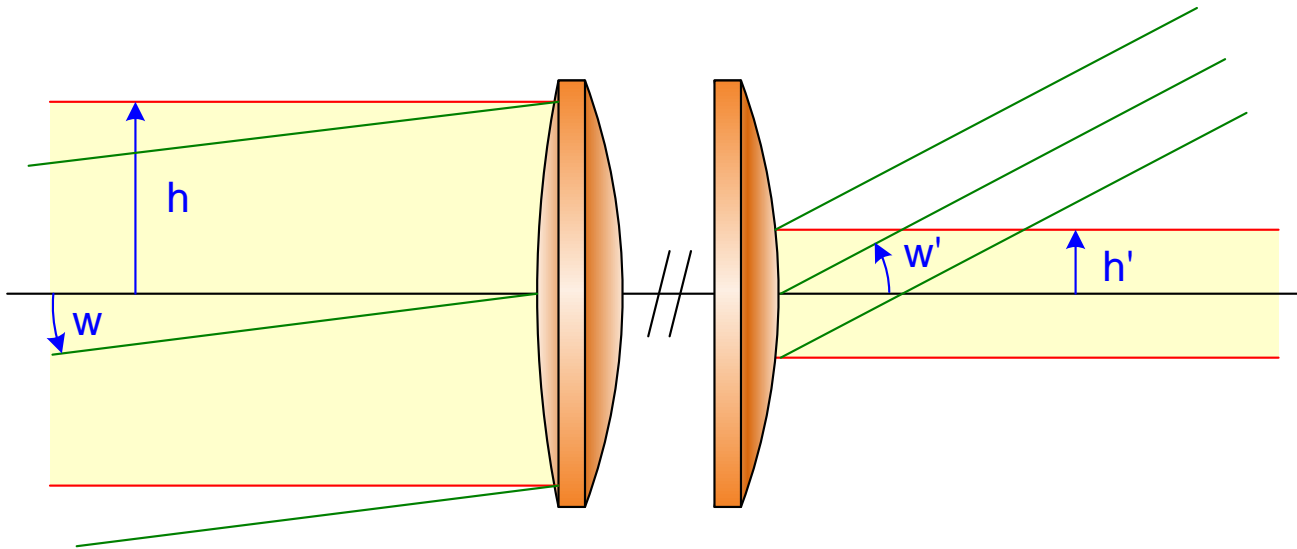




Angle Magnification

- Afocal systems with object/image in infinity
- Definition with field angle w
angular magnification

$$\Gamma = \frac{\tan w'}{\tan w} = \frac{nh}{n'h'}$$



- Relation with finite-distance magnification

$$m \cdot \Gamma = -\frac{f}{f'}$$

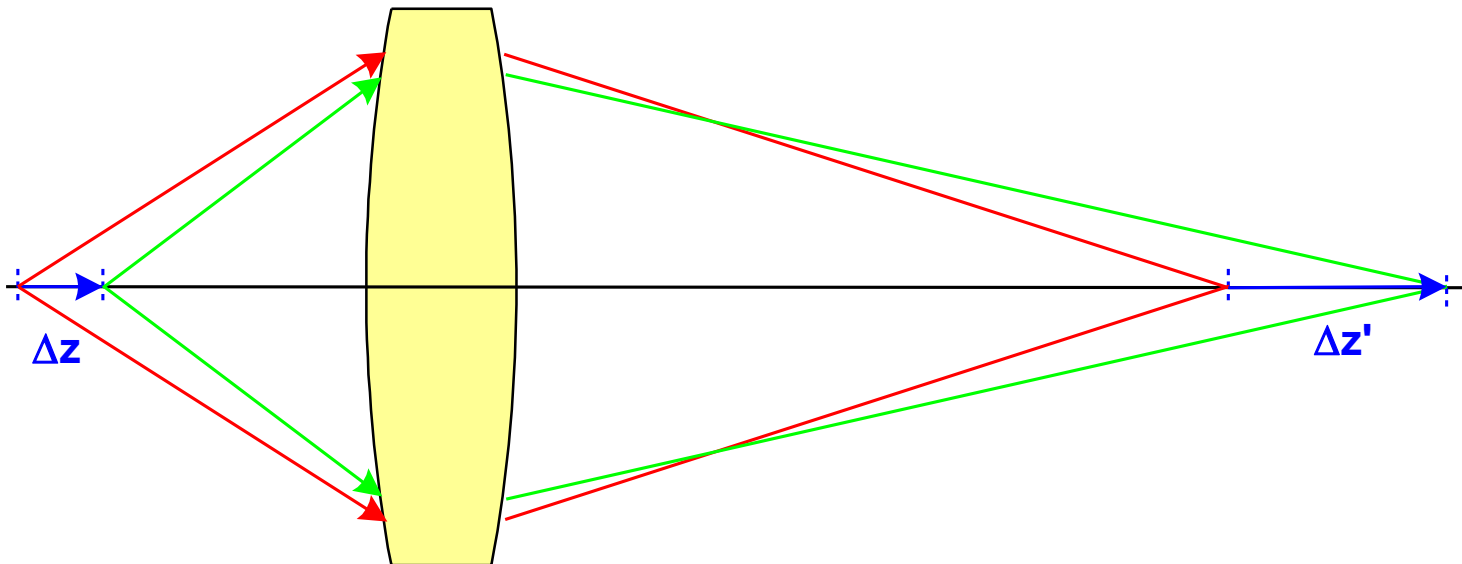


Axial Magnification

- Axial magnification
- Approximation for small Δz and $n = n'$

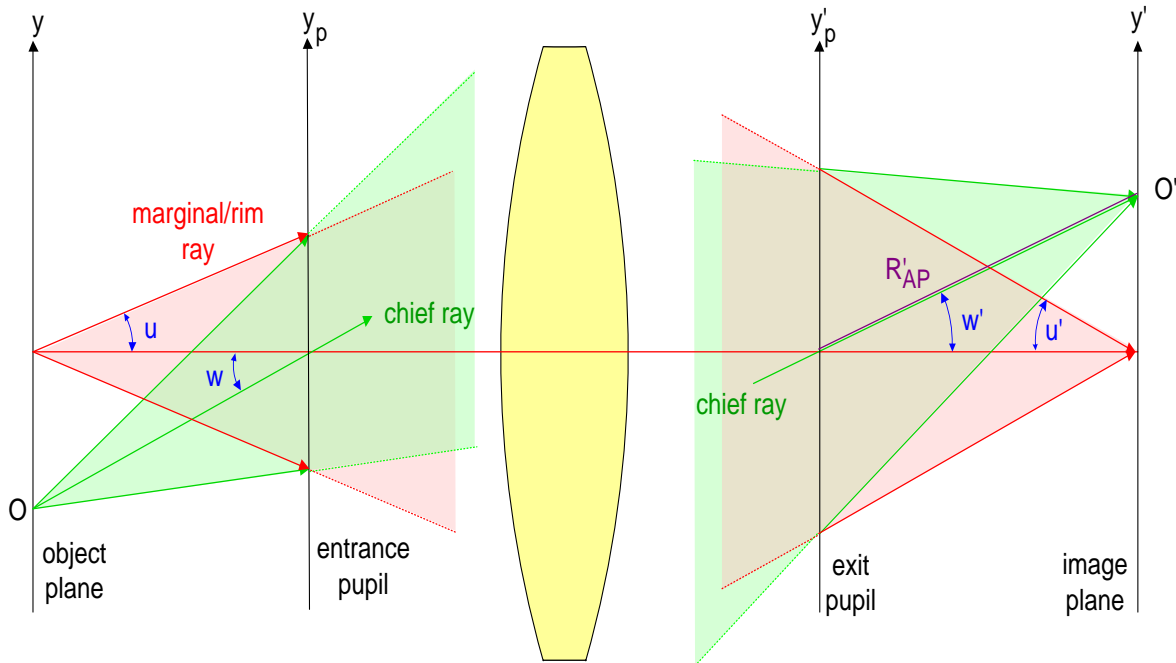
$$\alpha = \frac{\Delta z'}{\Delta z} = -m^2 \cdot \frac{f'}{f} \cdot \frac{1}{1 - \frac{m \cdot \Delta z}{f}}$$

$$\alpha = -m^2 = -\frac{\tan^2 u}{\tan^2 u'}$$

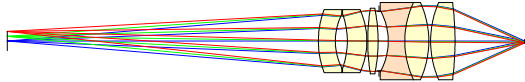
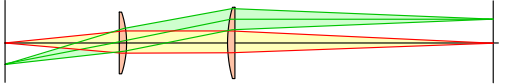
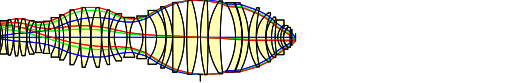
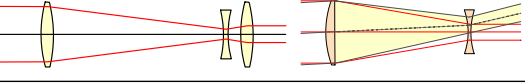
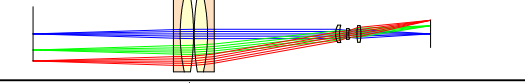
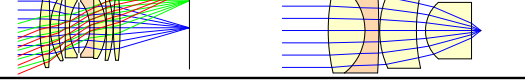
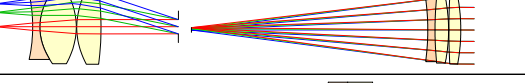
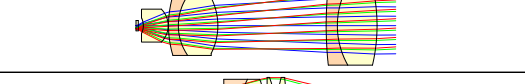
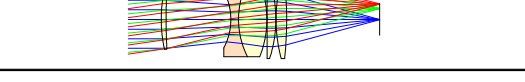


Definition of Field of View and Aperture

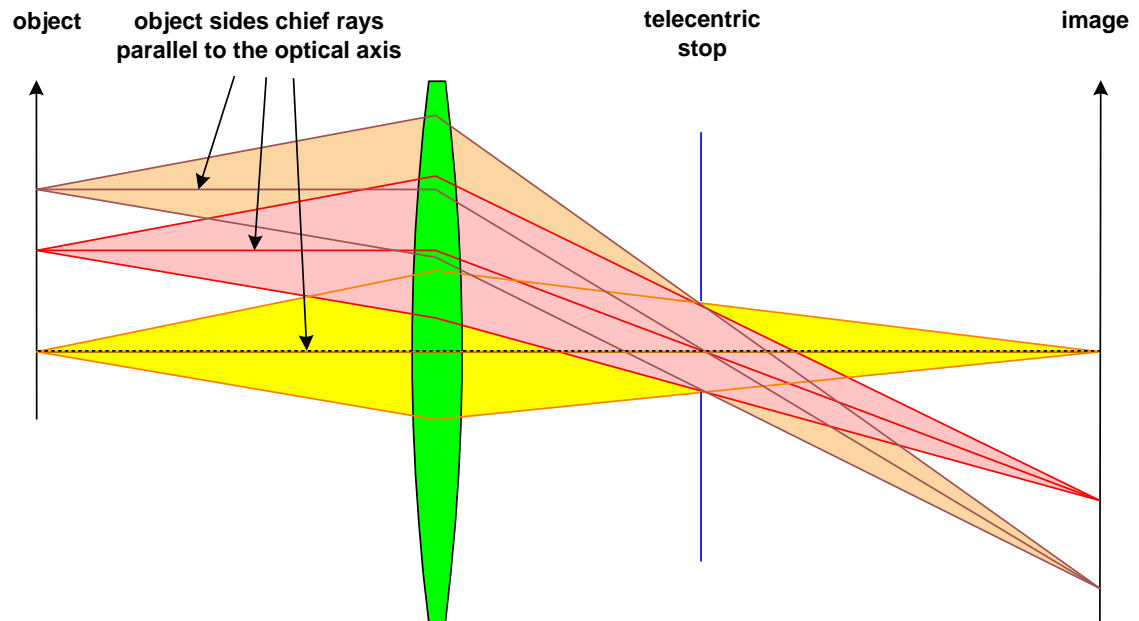
- Imaging on axis: circular / rotational symmetry
Only spherical aberration and chromatical aberrations
- Finite field size, object point off-axis:
 - chief ray as reference
 - skew ray bundles: coma and distortion
 - Vignetting, cone of ray bundle not circular symmetric
 - to distinguish: tangential and sagittal plane



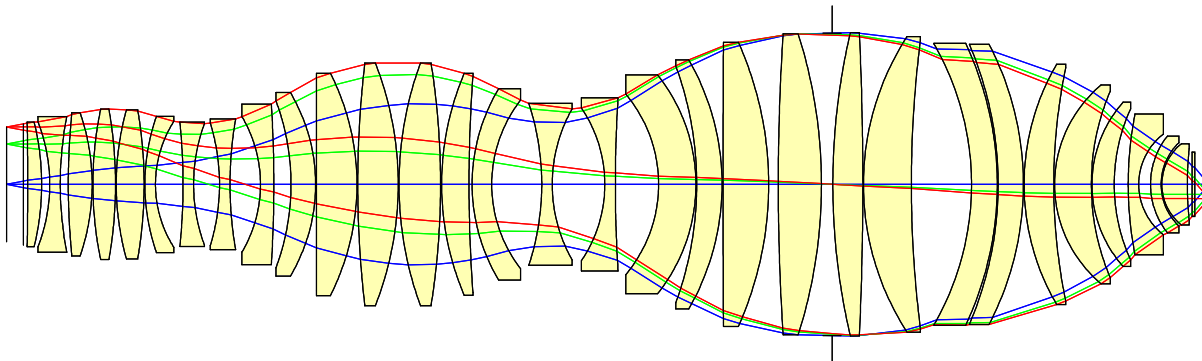
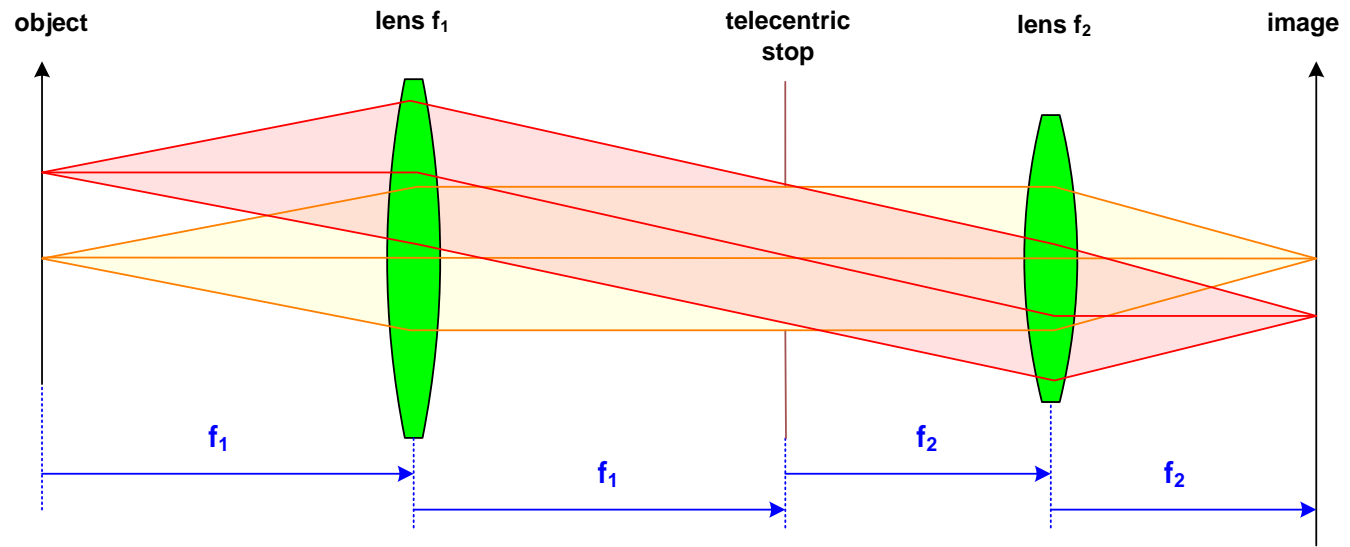
- Systematic of all infinity cases
- Physically impossible:
 1. object and entrance pupil in infinity
 2. image and exit pupil in infinity

case	object	image	entrance pupil	exit pupil	example	sample layout
1	finite	finite	finite	finite	relay	
2	finite	finite	finite	infinity image telecentric	metrology lens	
3	finite	finite	infinity object telecentric	infinity image telecentric	lithographic projection lens 4f-system	
4	infinity	infinity	finite	finite	afocal zoom telescopes beam expander	
5	finite	finite	infinity	finite	metrology lens	
6	infinity	finite	finite	finite	camera lens focussing lens	
7	finite	infinity	finite	finite	eyepiece collimator	
8	finite	infinity	infinity object telecentric	finite	microscopic lens	
9	infinity	finite	finite	infinity image telecentric	infinity metrology lens	
10	infinity	finite	infinity	finite	impossible	
11	finite	infinity	finite	infinity	impossible	
12	infinity	infinity	infinity	finite	impossible	
13	infinity	finite	infinity	infinity	impossible	
14	infinity	infinity	finite	infinity	impossible	
15	finite	infinity	infinity	infinity	impossible	
16	infinity	infinity	infinity	infinity	impossible	

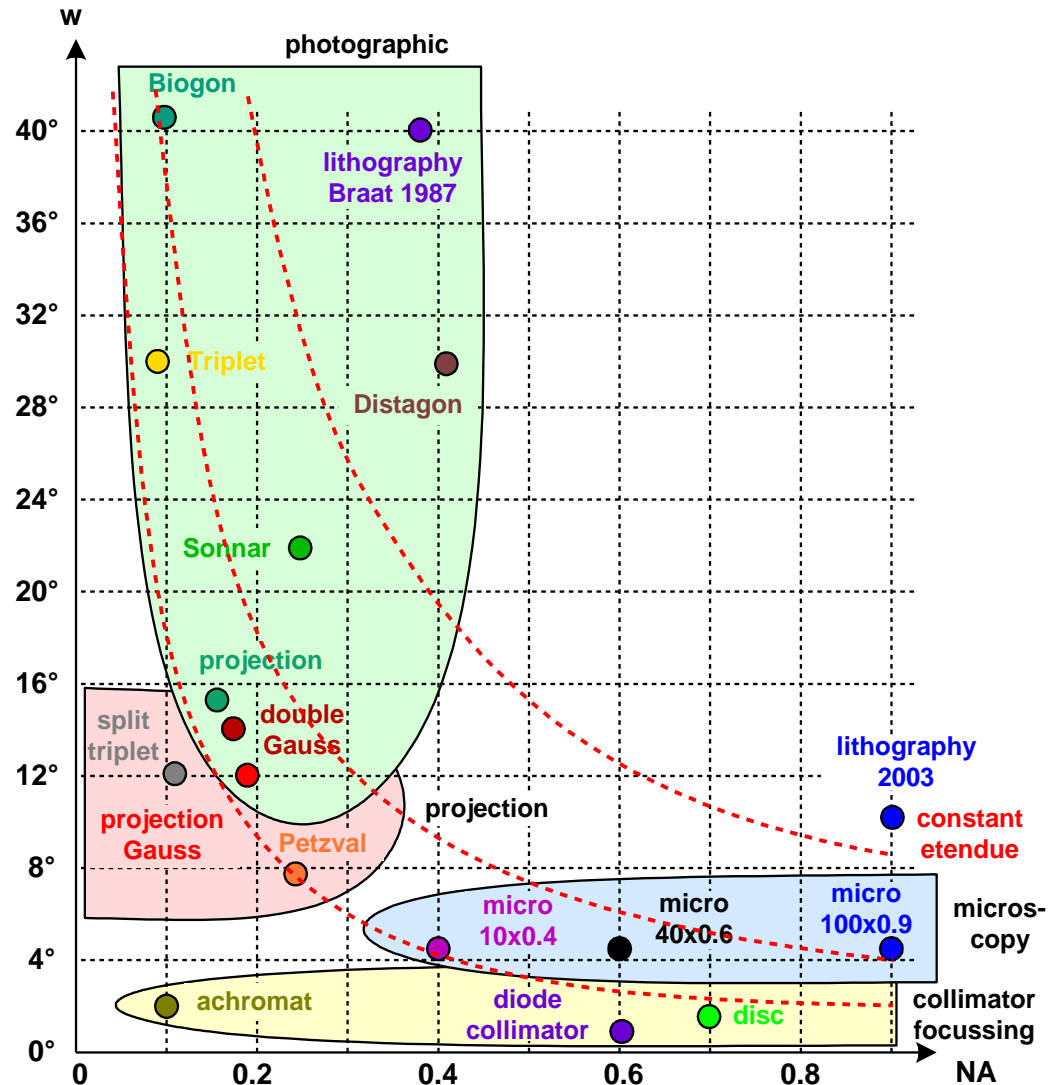
- Special stop positions:
 1. stop in back focal plane: object sided telecentricity
 2. stop in front focal plane: image sided telecentricity
 3. stop in intermediate focal plane: both-sided telecentricity
- Telecentricity:
 1. pupil in infinity
 2. chief ray parallel to the optical axis



- Double telecentric system: stop in intermediate focus
- Realization in lithographic projection systems

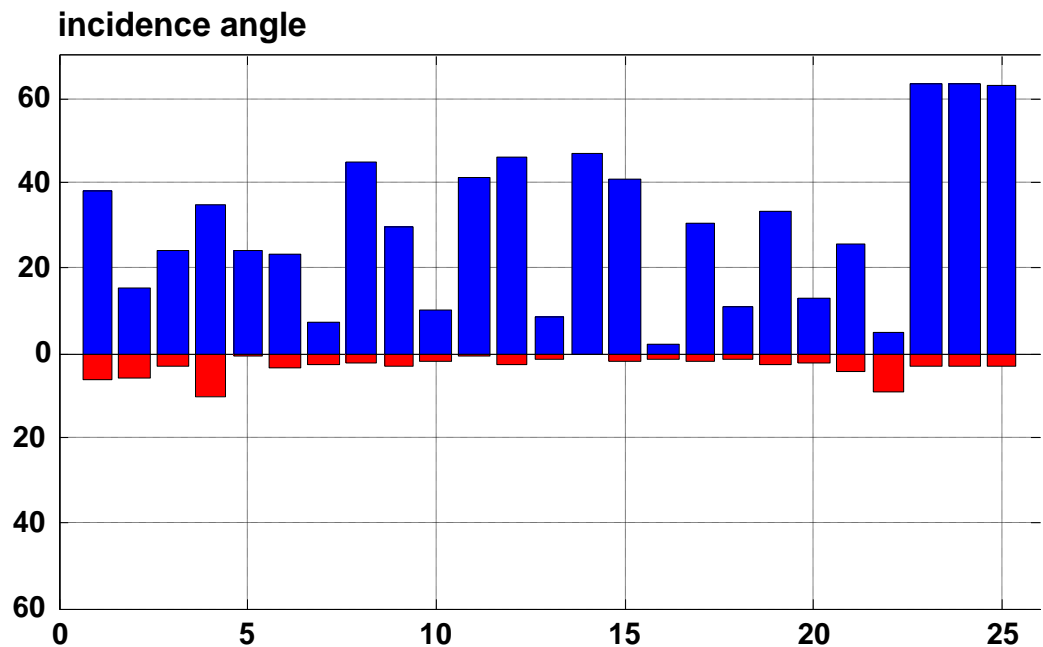
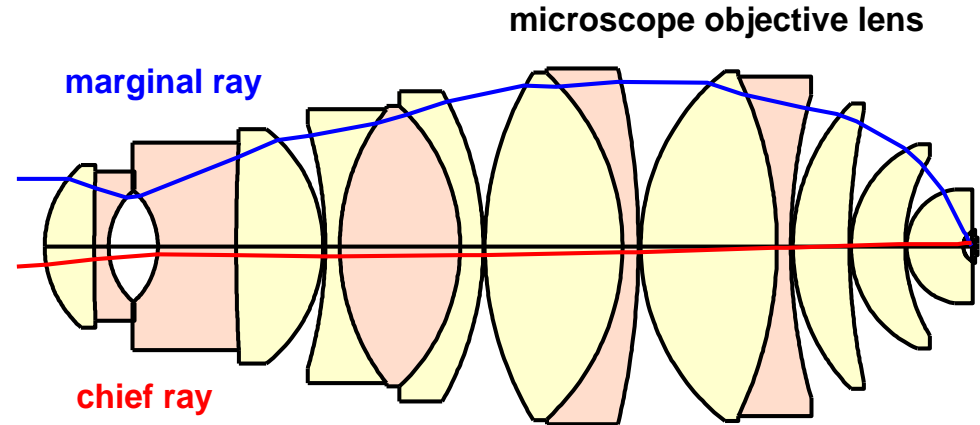


- Classification of systems with field and aperture size
- Scheme is related to size, correction goals and etendue of the systems
- Aperture dominated:
Disk lenses, microscopy,
Collimator
- Field dominated:
Projection lenses,
camera lenses,
Photographic lenses
- Spectral width as a correction requirement is missed in this chart

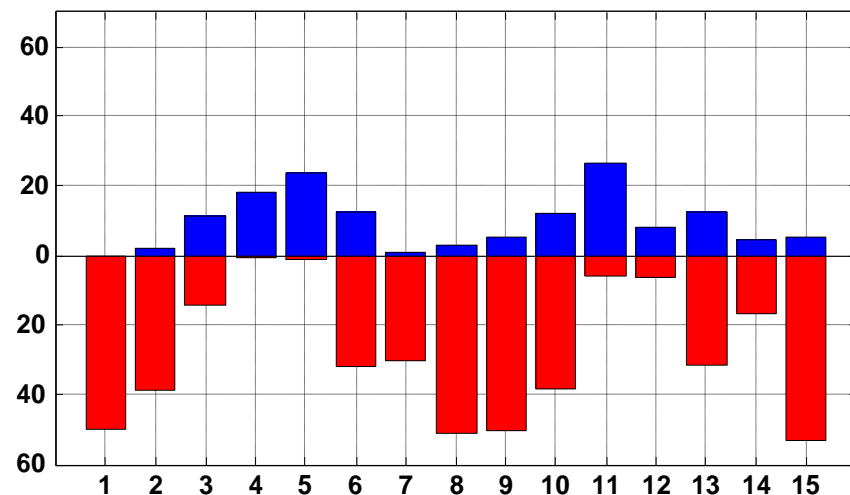
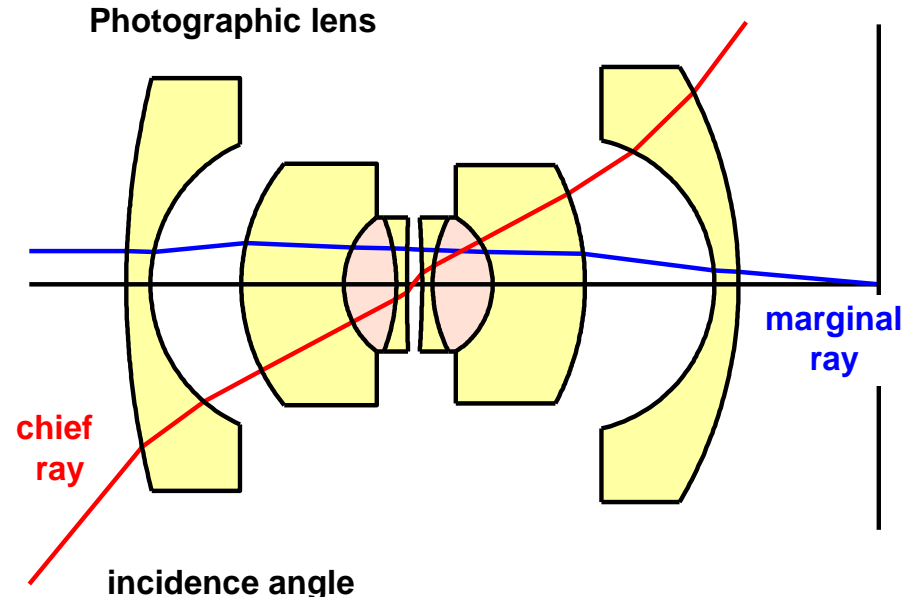


Microscopic Objective Lens

- Incidence angles for chief and marginal ray
- Aperture dominant system
- Primary problem is to correct spherical aberration



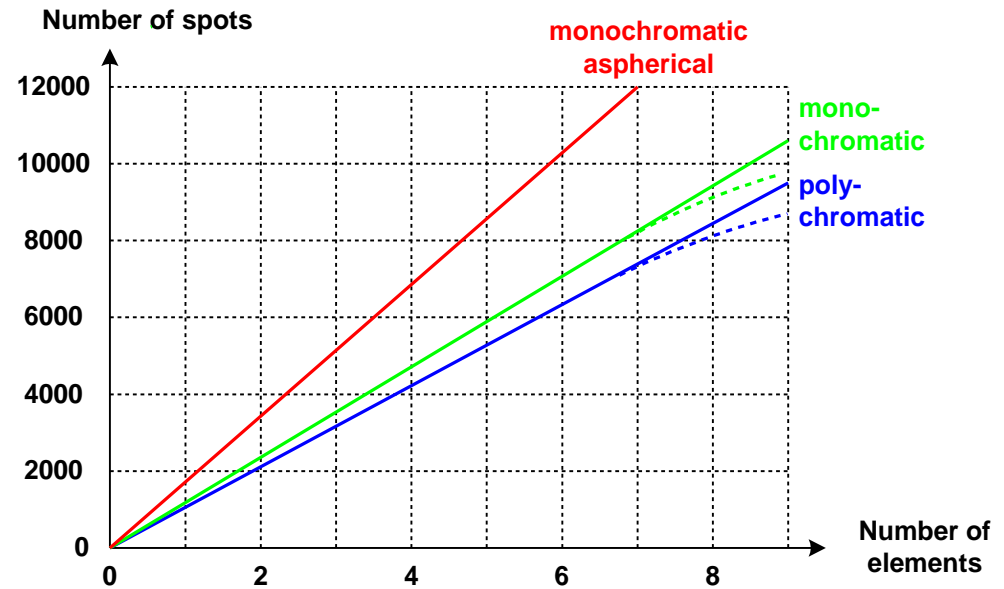
- Incidence angles for chief and marginal ray
- Field dominant system
- Primary goal is to control and correct field related aberrations: coma, astigmatism, field curvature, lateral color



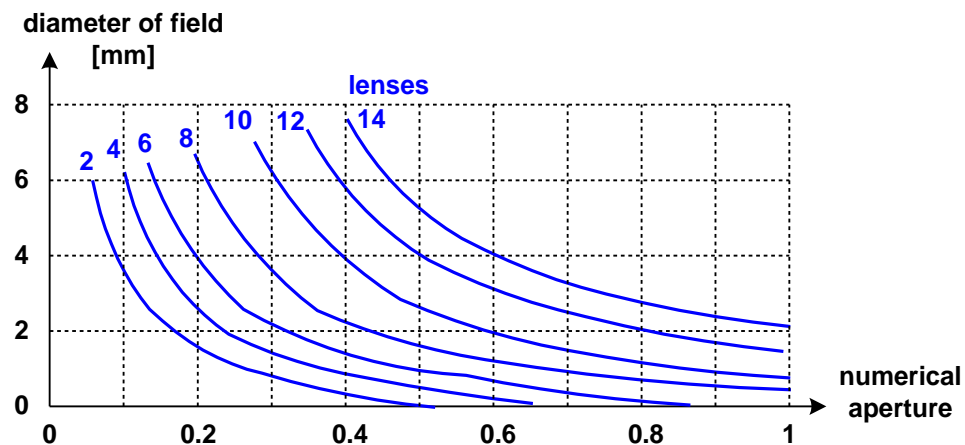
Number of Lenses



- Approximate number of spots over the field as a function of the number of lenses
Linear for small number of lenses.
Depends on mono-/polychromatic design and aspherics.



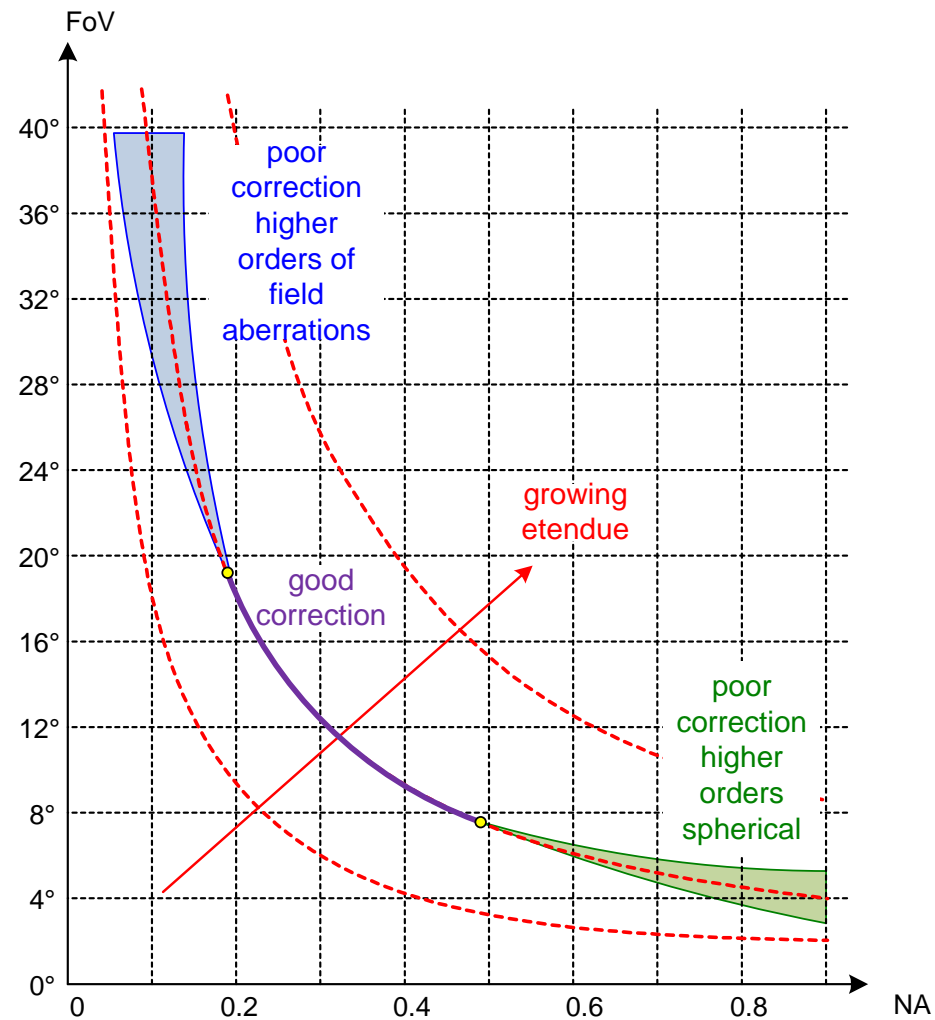
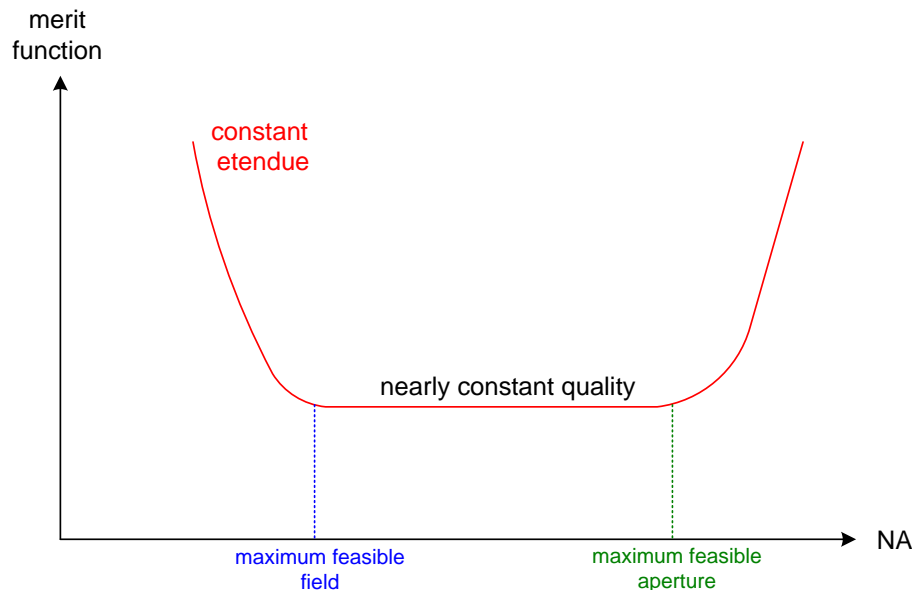
- Diffraction limited systems with different field size and aperture





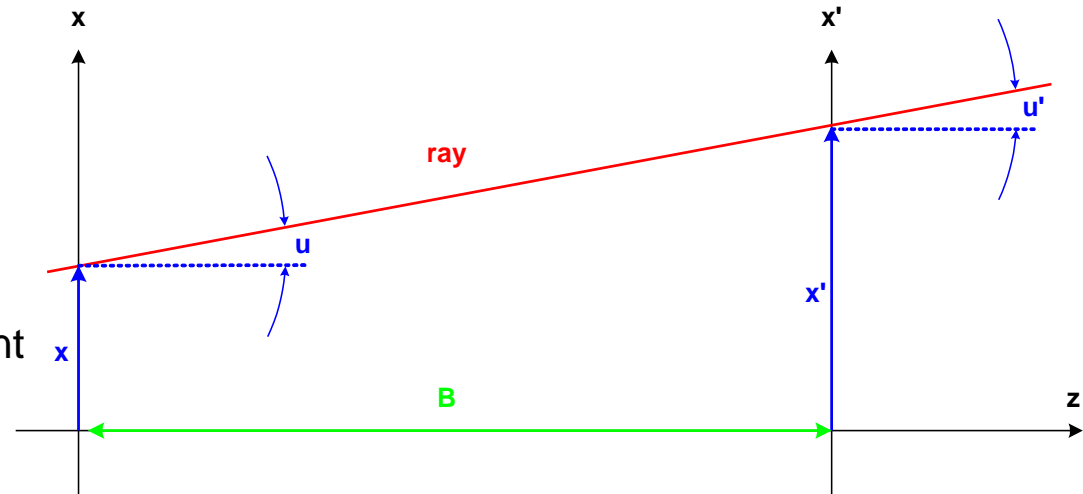
Foveated Imaging Range

- In the NA-FoV diagram hyperbolas are indicating constant etendue
- Correction in the central part are nearly equivalent and give a good correction
- In the extreme range of FoV or NA, large angles introduce non-correctable higher order aberrations
- This limits a feasible range of foveated imaging systems

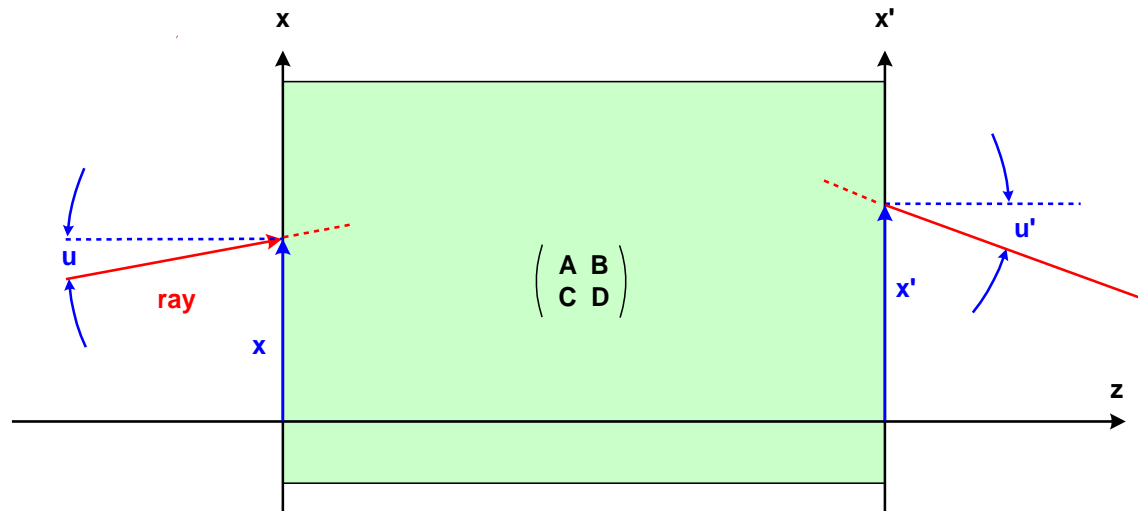


Matrix Formulation of Paraxial Optics

- Linear relation of ray transport
- Simple case: free space propagation
- Advantages of matrix calculus:
 1. simple calculation of component combinations
 2. Automatic correct signs of properties
 3. Easy to implement
- General case:
paraxial segment with matrix
ABCD-matrix :



$$\begin{pmatrix} x' \\ u' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x \\ u \end{pmatrix} = \underline{\underline{M}} \cdot \begin{pmatrix} x \\ u \end{pmatrix}$$





Matrix Calculus

- Paraxial raytrace transfer

$$y_j = y_{j-1} + d_{j-1} \cdot U_{j-1}$$

$$U_j' = U_{j-1}$$

- Matrix formulation

$$\begin{pmatrix} y_j' \\ U_j' \end{pmatrix} = \begin{pmatrix} 1 & d_{j-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_j \\ U_j \end{pmatrix}$$

- Matrix formalism for finite angles

$$\begin{pmatrix} y_j' \\ \tan u_j' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} y_j \\ \tan u_j \end{pmatrix}$$

- Paraxial raytrace refraction

$$y_j = y_{j-1} \quad i_j = \rho_j \cdot y_j + U_{j-1}$$

$$i_j' = \frac{n_j}{n_j'} i_j$$

$$U_j' = U_{j-1} - i_j + i_j'$$

- Inserted

$$U_j' = -\frac{\rho_j \cdot (n_j' - n_j)}{n_j} y_j + \frac{n_j}{n_j'} U_{j-1}$$

- Matrix formulation

$$\begin{pmatrix} y_j' \\ U_j' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\rho_j \cdot (n_j' - n_j)}{n_j} & \frac{n_j}{n_j'} \end{pmatrix} \cdot \begin{pmatrix} y_j \\ U_j \end{pmatrix}$$



Matrix Formulation of Paraxial Optics

- Linear transfer of spation coordinate x and angle u
- Matrix representation
- Lateral magnification for $u=0$
- Angle magnification of conjugated planes
- Refractive power for $u=0$
- Composition of systems
- Determinant, only 3 variables

$$x' = Ax + Bu$$

$$u' = Cx + Du$$

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x \\ u \end{pmatrix} = \underline{\underline{M}} \cdot \begin{pmatrix} x \\ u \end{pmatrix}$$

$$A = x'/x = \beta$$

$$D = u'/u = \gamma$$

$$C = u'/x$$

$$\underline{\underline{M}} = \underline{\underline{M}}_k \cdot \underline{\underline{M}}_{k-1} \cdot \dots \cdot \underline{\underline{M}}_2 \cdot \underline{\underline{M}}_1$$

$$\det \underline{\underline{M}} = AD - BC = \frac{n}{n'}$$



Matrix Formulation of Paraxial Optics

- System inversion

$$\underline{M}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

- Transition over distance L

$$\underline{M} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

- Thin lens with focal length f

$$\underline{M} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

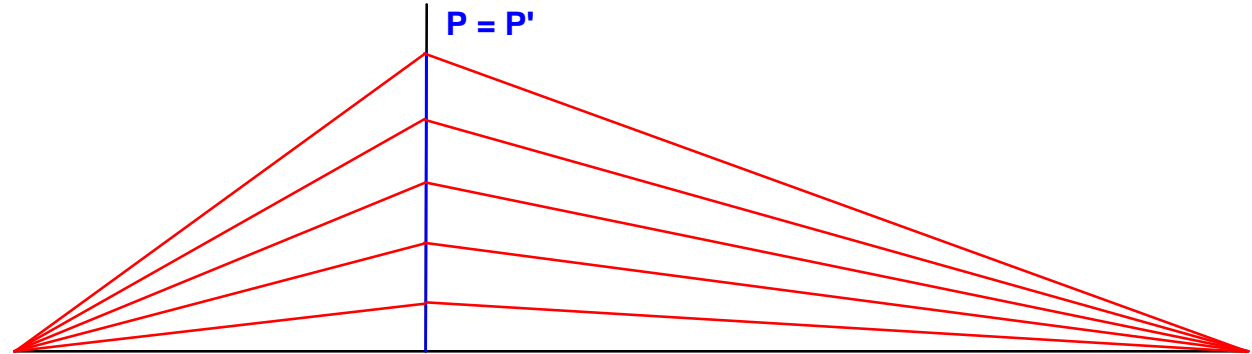
- Dielectric plane interface

$$\underline{M} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{pmatrix}$$

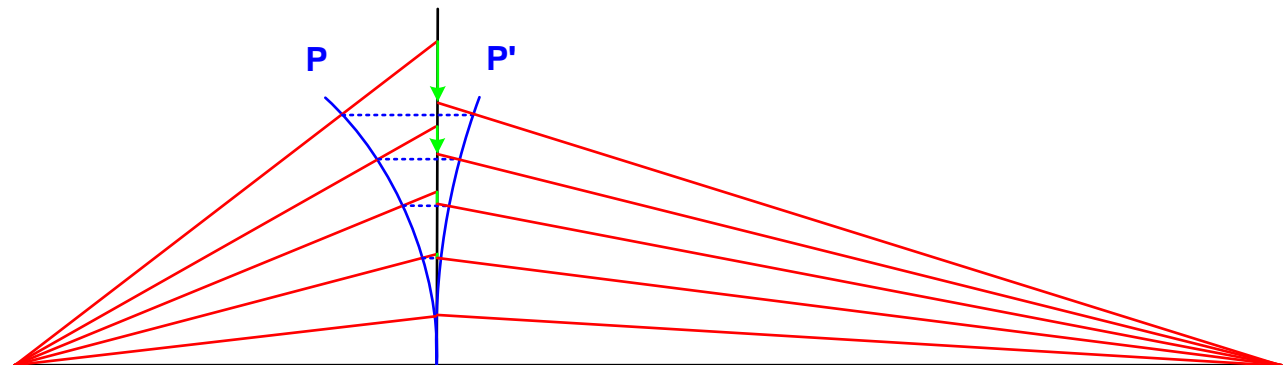
- Afocal telescope

$$\underline{M} = \begin{pmatrix} 1 & L \\ \frac{1}{\Gamma} & \Gamma \end{pmatrix}$$

- Ideal lens
 - one principal plane



- Aplanatic lens
 - principal surfaces are spheres
 - the marginal ray heights in the vortex plane are different for larger angles
 - inconsistencies in the layout drawings



What is ,Ideal' ?

- The notation ,ideal' imaging is not unique
- Ideal is in any case the location of the image point
- The geometrical ray paths can be different for
 1. paraxial
 2. ideal / linear collineation
 3. aplanatic
- The photometric properties are different due to non-equidistant sampling
- If a perfect lens is idealized in a software as one surface, there are principal discrepancies in the location of the intersection points

