



**Institute of  
Applied Physics**

Friedrich-Schiller-Universität Jena

# Imaging and Aberration Theory

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Lecture 4: Aberration expansions

2018-11-09

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# Schedule - Imaging and aberration theory 2018

1	19.10.	Paraxial imaging	paraxial optics, fundamental laws of geometrical imaging, compound systems
2	26.10.	Pupils, Fourier optics, Hamiltonian coordinates	pupil definition, basic Fourier relationship, phase space, analogy optics and mechanics, Hamiltonian coordinates
3	02.11.	Eikonal	Fermat principle, stationary phase, Eikonals, relation rays-waves, geometrical approximation, inhomogeneous media
4	09.11.	Aberration expansions	single surface, general Taylor expansion, representations, various orders, stop shift formulas
5	16.11.	Representation of aberrations	different types of representations, fields of application, limitations and pitfalls, measurement of aberrations
6	23.11.	Spherical aberration	phenomenology, sph-free surfaces, skew spherical, correction of sph, aspherical surfaces, higher orders
7	30.11.	Distortion and coma	phenomenology, relation to sine condition, aplanatic systems, effect of stop position, various topics, correction options
8	07.12.	Astigmatism and curvature	phenomenology, Coddington equations, Petzval law, correction options
9	14.12.	Chromatical aberrations	Dispersion, axial chromatical aberration, transverse chromatical aberration, spherochromatism, secondary spectrum
10	21.12.	Sine condition, aplanatism and isoplanatism	Sine condition, isoplanatism, relation to coma and shift invariance, pupil aberrations, Herschel condition, relation to Fourier optics
11	11.01.	Wave aberrations	definition, various expansion forms, propagation of wave aberrations
12	18.01.	Zernike polynomials	special expansion for circular symmetry, problems, calculation, optimal balancing, influence of normalization, measurement
13	25.01.	Point spread function	ideal psf, psf with aberrations, Strehl ratio
14	01.02.	Transfer function	transfer function, resolution and contrast
15	08.02.	Additional topics	Vectorial aberrations, generalized surface contributions, Aldis theorem, intrinsic and induced aberrations, reverbability



- Repetition...
- Single refracting surface
- Adaptation on optical system terms
- Primary monochromatic aberration surface contributions
- Extension on several surfaces
- Lens contributions
- Examples

- Fermat principle: light takes the fastest way between two points  
The variance of the OPL is zero, stationary point
- The OPL is the Lagrange function in optics  
the equation of motion is
- This is the Eikonal equation of geometrical optics

$$\delta L = \delta \int_{P_1}^{P_2} n(x, y, z) ds = 0$$

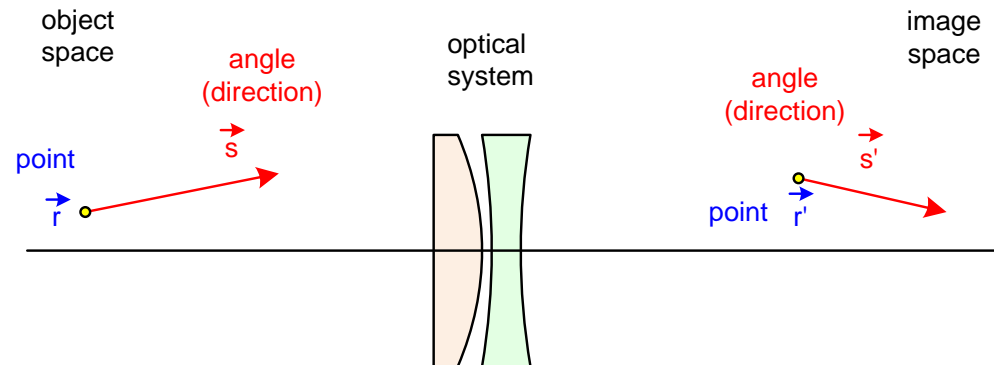
$$\frac{\partial L}{\partial x'} = n \cdot s_x, \quad \frac{\partial L}{\partial y'} = n \cdot s_y$$

$$\frac{d}{ds} \left( n \cdot \frac{d\vec{r}}{ds} \right) = \nabla n$$

- One ray is fixed by 4 parameters
- The general 4x4 functional relation between object and image space is restricted by the Fermat principle: only 4 independent variables
- A Legendre transform allows to switch between the variables

$$L(x, y, s_x, s_y) = n s_x \cdot x' + n s_y \cdot y' - L(x, y, x', y')$$

- Every selection of eikonal variables has at least one case, which is singular



- Point eikonal: spatial coordinates in object and image space are fixed the corresponding angles can be calculated by the differential equations

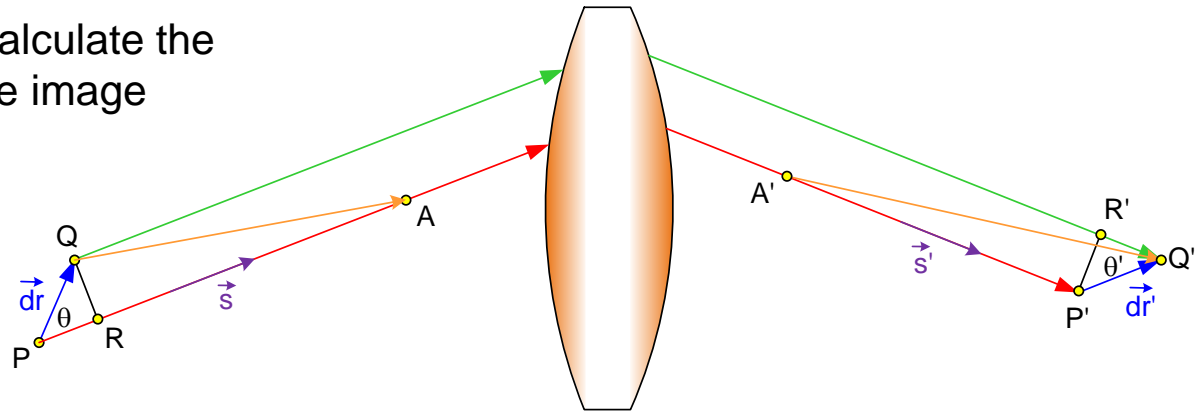
$$dL_P(x, y, x', y') = n' \cdot (s'_x dx' + s'_y dy') - n \cdot (s_x dx + s_y dy)$$

$$\frac{\partial L_P}{\partial x} = -ns_x \quad \frac{\partial L_P}{\partial y} = -ns_y$$

$$\frac{\partial L_P}{\partial x'} = n's'_x \quad \frac{\partial L_P}{\partial y'} = n's'_y$$

- Small change of initial ray data in the object space:  
Fermat principle allows to calculate the corresponding change in the image space

$$\delta L = n' \vec{s}' \cdot d\vec{r}' - n \vec{s} \cdot d\vec{r}$$



- Special case  $\theta = 90^\circ$  delivers the Abbe sine condition

$$m = \frac{n \sin u}{n' \sin u'} = \frac{y'}{y}$$



# Refracting Surface I: Basic Eikonal Approach

- P, P' points on ray
- A, A' arbitrary points on axis
- s, s' direction unit vectors of the rays relationship
- Real surface equation contains system parameters R, b

$$s_z = \sqrt{1 - s_x^2 - s_y^2} \quad , \quad s'_z = \sqrt{1 - s'^2_x - s'^2_y}$$

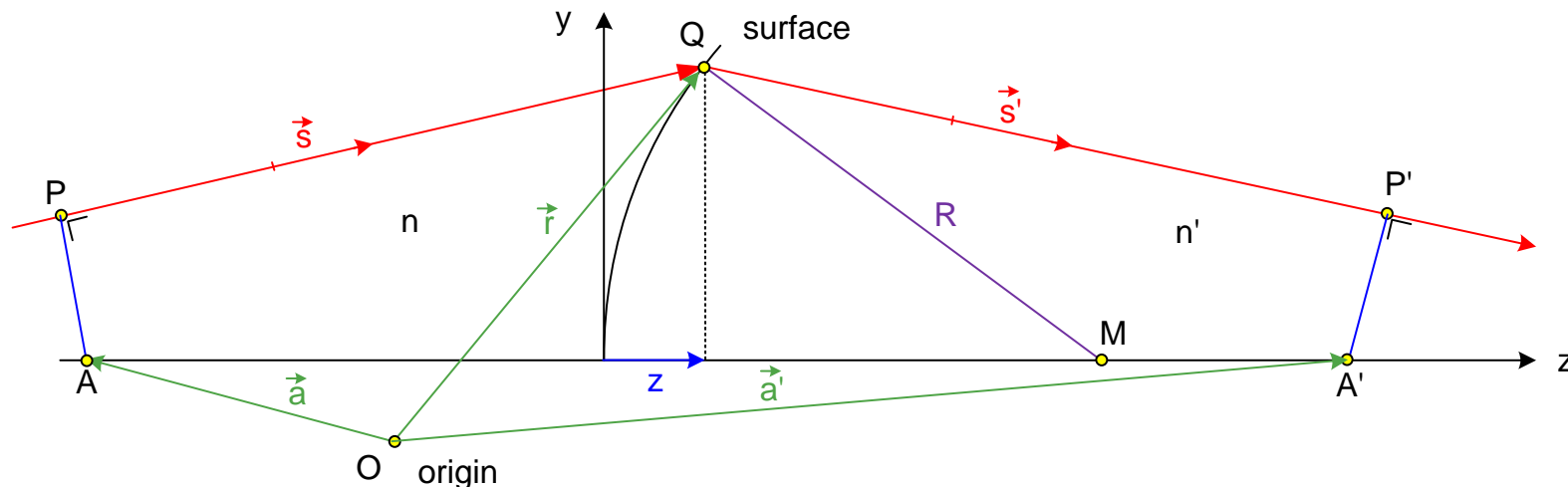
$$z = \frac{x^2 + y^2}{2R} + \frac{(x^2 + y^2)^2}{8R^3} \cdot (1 + b) + \dots$$

- Angle eikonal, gives the optical path length change in P', if the point P is varied

$$dL_A(\vec{s}, \vec{s}') = n \cdot (\vec{r} - \vec{a}) \cdot d\vec{s} - n' \cdot (\vec{r}' - \vec{a}') \cdot d\vec{s}'$$

In coordinate representation

$$dL_A = n \cdot [x \cdot ds_x + y \cdot ds_y + (z - a) \cdot ds_z] - n' \cdot [x \cdot ds'_x + y \cdot ds'_y + (z - a') \cdot ds'_z]$$





# Refracting Surface II: Eikonal

- Integration of the differential to get the total optical path length

$$dL_A = n \cdot [x \cdot ds_x + y \cdot ds_y + (z - a) \cdot ds_z] - n' \cdot [x \cdot ds'_x + y \cdot ds'_y + (z - a') \cdot ds'_z]$$

- Result of 4th order Taylor approximation

$$\begin{aligned} L_A = & -n \cdot a + n' \cdot a' \\ & + n \cdot \left[ x \cdot s_x + y \cdot s_y + \frac{x^2 + y^2}{2R} + \frac{a}{2} \cdot (s_x^2 + s_y^2) \right] \\ & - n' \cdot \left[ x \cdot s'_x + y \cdot s'_y + \frac{x^2 + y^2}{2R} + \frac{a'}{2} \cdot (s'^2_x + s'^2_y) \right] \\ & + n \cdot \left[ \frac{1+b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s_x^2 + s_y^2)}{4R} + \frac{a}{8} \cdot (s_x^2 + s_y^2)^2 \right] \\ & - n' \cdot \left[ \frac{1+b}{8R^3} \cdot (x^2 + y^2)^2 + \frac{(x^2 + y^2) \cdot (s'^2_x + s'^2_y)}{4R} + \frac{a'}{8} \cdot (s'^2_x + s'^2_y)^2 \right] \end{aligned}$$

- Elimination of intermediate variables  $x, y$  of the intersection point with the help of the law of refraction



# Refracting Surface III: Law of Refraction

- Implicite surface equation

$$F(x, y, z) = -z + \frac{x^2 + y^2}{2R} + \frac{(1+b) \cdot (x^2 + y^2)^2}{8R^3} = 0$$

- Normal unit vector in the intersection point

$$\vec{e} = \nabla F(x, y, z)$$

$$e_x = \frac{x}{R} \cdot \left[ 1 + \frac{(1+b) \cdot (x^2 + y^2)}{2R^2} \right], \quad e_y = \frac{y}{R} \cdot \left[ 1 + \frac{(1+b) \cdot (x^2 + y^2)}{2R^2} \right], \quad e_z = -1$$

- Law of refraction

$$(n \cdot \vec{s} - n' \vec{s}') \times \vec{e} = 0$$

- Formulation in components

$$-(ns_y - n's'_y) - (ns_z - n's'_z) \cdot \frac{y}{R} \cdot \left[ 1 + \frac{(1+b) \cdot (x^2 + y^2)}{2R^2} \right] = 0$$

$$+(ns_x - n's'_x) + (ns_z - n's'_z) \cdot \frac{x}{R} \cdot \left[ 1 + \frac{(1+b) \cdot (x^2 + y^2)}{2R^2} \right] = 0$$

$$(ns_x - n's'_x) \cdot \frac{y}{R} \cdot \left[ 1 + \frac{(1+b) \cdot (x^2 + y^2)}{2R^2} \right] - (ns_y - n's'_y) \cdot \frac{x}{R} \cdot \left[ 1 + \frac{(1+b) \cdot (x^2 + y^2)}{2R^2} \right] = 0$$

- Solving for x, y in approximation of 4th order

$$x = -R \cdot \frac{ns_x - n's'_x}{n - n'}, \quad y = -R \cdot \frac{ns_y - n's'_y}{n - n'}$$





# Refracting Surface IV: Eikonal

- Resulting Eikonal function of  $s_x, s_y, s'_x, s'_y$  only

$$\begin{aligned}
 L_A &= L_A^{(0)} + L_A^{(2)} + L_A^{(4)} \\
 &= na - n'a' - \frac{R}{2(n-n')} \cdot \left[ (ns_x - n's'_x)^2 + (ns_y - n's'_y)^2 + \frac{na}{2} \cdot (s_x^2 + s_y^2) - \frac{n'a'}{2} \cdot (s'^2_x + s'^2_y) \right] \\
 &\quad - \frac{R}{4(n-n')^2} \cdot \left[ (ns_x - n's'_x)^2 + (ns_y - n's'_y)^2 \right] \cdot \left[ +n \cdot (s_x^2 + s_y^2) - n' \cdot (s'^2_x + s'^2_y) \right] \\
 &\quad + \frac{1}{8} \cdot \left[ na \cdot (s_x^2 + s_y^2)^2 - n'a' \cdot (s'^2_x + s'^2_y)^2 \right] + \frac{(1+b)R}{8(n-n')^3} \cdot \left[ (ns_x - n's'_x)^2 + (ns_y - n's'_y)^2 \right]
 \end{aligned}$$

- Further re-arrangements for better practical usage:

1. introduction of the pupil coordinates  $x_p, y_p$  instead of the image sides directions  $s'_x, s'_y$
2. switch to the image space ray parameters
3. optional switch to circular coordinates
4. according to Seidel substitution of the system parameter by the paraxial properties of the system and the marginal ray
5. Further approximation in 4th order to get a perturbation representation of the paraxial imaging. This allows a decoupling of the surface contributions



# Refracting Surface V: Paraxial Optics

- 2nd order: paraxial optics

$$x = \frac{1}{n} \frac{\partial L_A^{(2)}}{\partial s_x} = s_x \cdot \left( a - \frac{nR}{n-n'} \right) + s'_x \cdot \frac{n'R}{n-n'}$$

$$y = \frac{1}{n} \frac{\partial L_A^{(2)}}{\partial s_y} = s_y \cdot \left( a - \frac{nR}{n-n'} \right) + s'_y \cdot \frac{n'R}{n-n'}$$

$$x' = -\frac{1}{n'} \frac{\partial L_A^{(2)}}{\partial s'_x} = s'_x \cdot \left( a' + \frac{n'R}{n-n'} \right) - s_x \cdot \frac{nR}{n-n'}$$

$$y' = -\frac{1}{n'} \frac{\partial L_A^{(2)}}{\partial s'_y} = s'_y \cdot \left( a' + \frac{n'R}{n-n'} \right) - s_y \cdot \frac{nR}{n-n'}$$

- Eliminating  $s'_x$ ,  $s'_y$

$$x' = x \cdot \frac{a'}{n'} \cdot \left( \frac{n'}{a'} + \frac{n-n'}{R} \right) - \frac{aa'}{n'} s_x \cdot \left( \frac{n'}{a'} - \frac{n}{a} + \frac{n-n'}{R} \right)$$

$$y' = y \cdot \frac{a'}{n'} \cdot \left( \frac{n'}{a'} + \frac{n-n'}{R} \right) - \frac{aa'}{n'} s_y \cdot \left( \frac{n'}{a'} - \frac{n}{a} + \frac{n-n'}{R} \right)$$

- Imaging condition:  $x'$ ,  $y'$  independent from ray directions  $s_x$ ,  $s_y$

1. imaging equation

$$\frac{n}{a} - \frac{n'}{a'} = \frac{n-n'}{R}$$

2. magnification

$$x' = x \cdot \frac{na'}{n'a} \quad , \quad m = \frac{x'}{x} = \frac{na'}{n'a}$$

# Refracting Surface VI: Pupil Coordinates

- Change to pupil coordinates for optical systems

- Relations

$$\tan u_x = \frac{x - x_p}{p} = \frac{s_x}{s_z}, \quad s_x = \frac{x - x_p}{p} \cdot s_z$$

$$\tan u_y = \frac{y - y_p}{p} = \frac{s_y}{s_z}, \quad s_y = \frac{y - y_p}{p} \cdot s_z$$

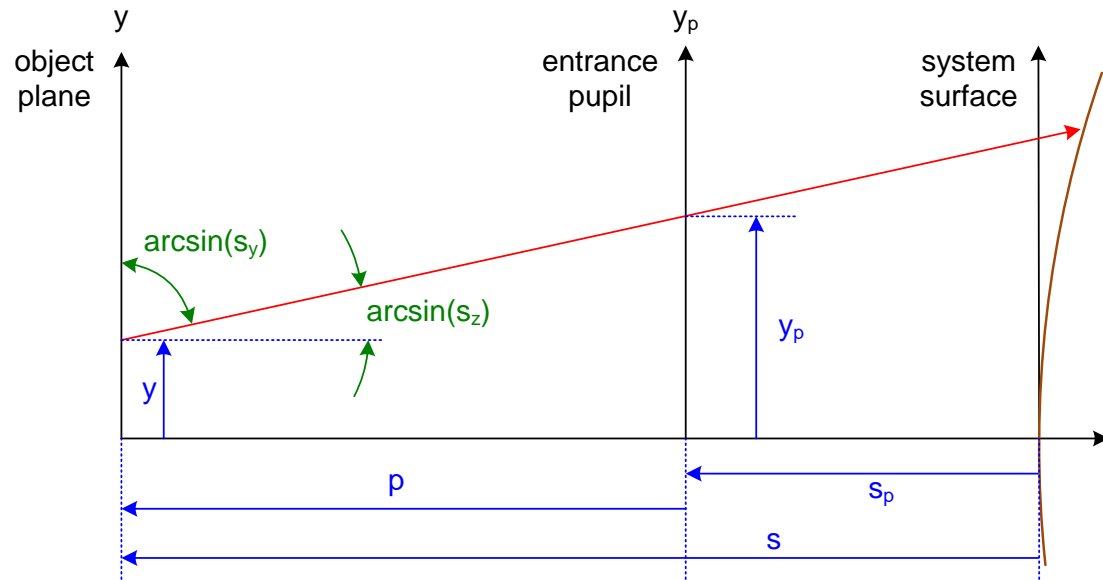
$$\tan u'_x = \frac{x' - x'_p}{p'} = \frac{s'_x}{s'_z}, \quad s'_x = \frac{x' - x'_p}{p'} \cdot s'_z$$

$$\tan u'_y = \frac{y' - y'_p}{p'} = \frac{s'_y}{s'_z}, \quad s'_y = \frac{y' - y'_p}{p'} \cdot s'_z$$

- Approximated to 2nd order

$$s_x = \frac{x - x_p}{p}, \quad s_y = \frac{y - y_p}{p}$$

$$s'_x = \frac{x' - x'_p}{p'}, \quad s'_y = \frac{y' - y'_p}{p'}$$





# Refracting Surface VII: Eikonal

## ▪ Eikonal of 4th order

$$L_A^{(4)} = K \cdot \frac{(s-p)^4}{8p^4} (x^2 + y^2)^2 + S \cdot \frac{s^4}{8p^4} (x_p^2 + y_p^2)^2 + A \cdot \frac{s^2 \cdot (s-p)^2}{2p^4} (xx_p + yy_p)^2 \\ + P \cdot \frac{s^2 \cdot (s-p)^2}{4p^4} (x^2 + y^2)(xx_p + yy_p) - D \cdot \frac{s \cdot (s-p)^3}{2p^4} (x^2 + y^2)(xx_p + yy_p) - C \cdot \frac{s^3 \cdot (s-p)}{2p^4} (x^2 + y^2)(xx_p + yy_p)$$

## ▪ Coefficients

### 1. Spherical aberration

$$K = -\frac{(n'-n)b}{R^3} - ns \cdot \left( \frac{1}{Rs} - \frac{1}{s_p^2} \right)^2 + n's' \cdot \left( \frac{1}{Rs'} - \frac{1}{s_p'^2} \right)^2$$

### 2. Astigmatism

$$S = -\frac{(n'-n)b}{R^3} - Q^2 \cdot \left( \frac{1}{ns} - \frac{1}{n's'} \right)$$

### 3. Field curvature

$$A = -\frac{(n'-n)b}{R^3} - Q_p^2 \cdot \left( \frac{1}{ns} - \frac{1}{n's'} \right)$$

### 4. Distortion

$$P = -\frac{(n'-n)b}{R^3} - QQ_p \cdot \left( \frac{1}{ns} - \frac{1}{n's'} \right) + Q(Q - Q_p) \cdot \left( \frac{1}{ns_p} - \frac{1}{n's_p'} \right)$$

### 5. Coma

$$D = -\frac{(n'-n)b}{R^3} - Q_p^2 \cdot \left( \frac{1}{ns} - \frac{1}{n's'} \right) + Q_p(Q - Q_p) \cdot \left( \frac{1}{ns_p} - \frac{1}{n's_p'} \right)$$

$$C = -\frac{(n'-n)b}{R^3} - QQ_p \cdot \left( \frac{1}{ns} - \frac{1}{n's'} \right)$$



# Refracting Surface VIII: Aberrations

- Corresponding differential equations of the angle eikonal determine the changes in the spatial coordinates
- Calculation of the transverse aberrations:  
separation of the paraxial and the perturbation part

$$x' = m \cdot x + \Delta x' = -\frac{1}{n'} \cdot \frac{\partial L_A}{\partial s_x'} \quad , \quad y' = m \cdot y + \Delta y' = -\frac{1}{n'} \cdot \frac{\partial L_A}{\partial s_y'}$$

$$\Delta x' = -\frac{1}{n'} \cdot \frac{\partial L_A^{(4)}}{\partial s_x'} \quad , \quad \Delta y' = -\frac{1}{n'} \cdot \frac{\partial L_A^{(4)}}{\partial s_y'}$$

- Used paraxial abbreviations:  
1. pupil imaging magnification

$$m_p = \frac{y'_p}{y_p} = \frac{s \cdot p'}{s' \cdot p}$$

- 2. Abbe invariants

$$Q = n \cdot \left( \frac{1}{R} - \frac{1}{s} \right) \quad , \quad Q_p = n \cdot \left( \frac{1}{R} - \frac{1}{s_p} \right)$$

- 3. usual special Picht-operator: difference before / after the refraction surface

$$\Delta(n) = n - n' \quad , \quad \Delta\left(\frac{n}{s}\right) = \frac{n}{s} - \frac{n'}{s'} \quad , \quad , \dots$$



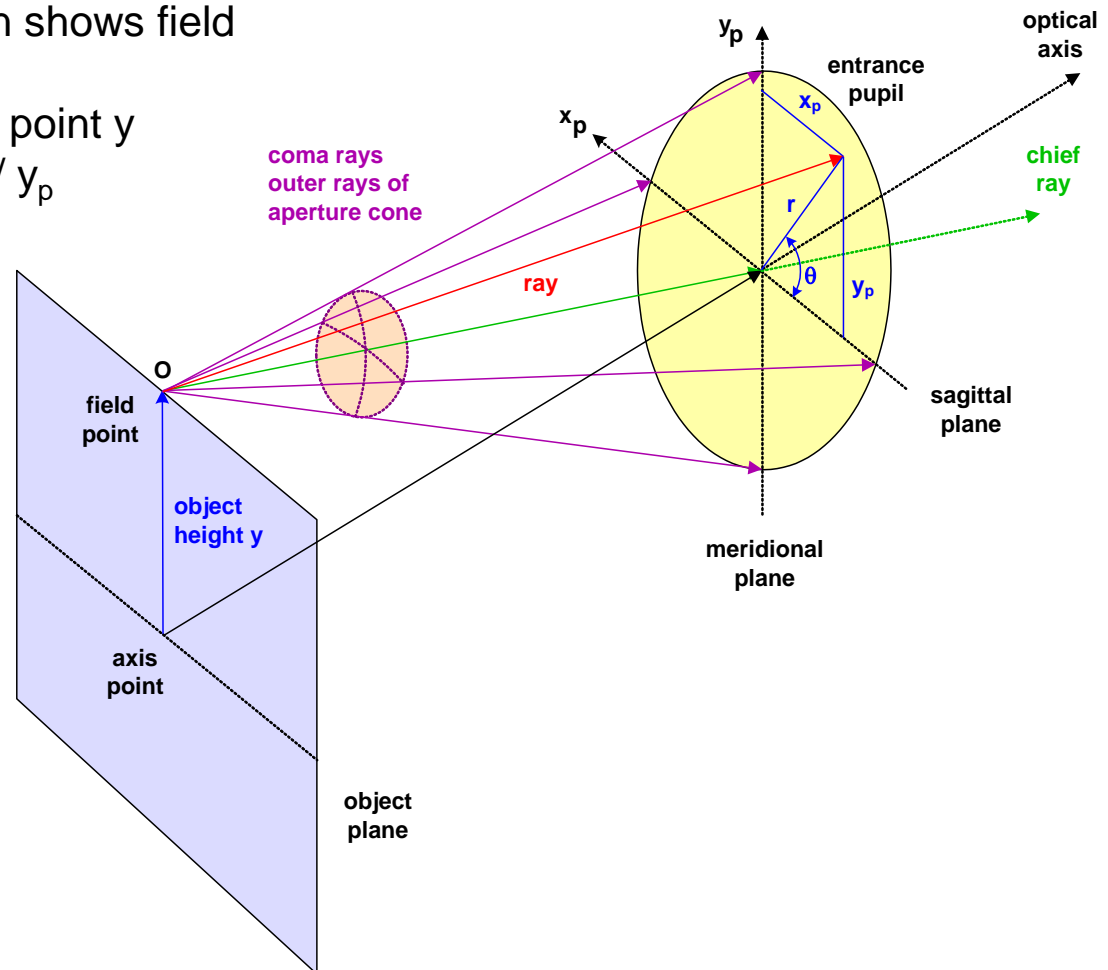
# Refracting Surface IX: Transverse Aberrations

- Transverse aberrations

$$\begin{aligned}\Delta x' &= S \cdot \frac{x'_p s'^4}{2n' p'^3} (x'^2_p + y'^2_p) + P \cdot \frac{x'_p s'^2 s'^2_p}{2n' p'^3} (x'^2 + y'^2) - C \cdot \frac{x'_p s'^3 s'_p}{n' p'^3} (x' x'_p + y' y'_p) \\ &\quad + A \cdot \frac{x' s'^3 s'_p}{2n' p'^3} (x'^2_p + y'^2_p) + D \cdot \frac{x' s' s'^3_p}{2n' p'^3} (x'^2 + y'^2) - C \cdot \frac{x' s'^2 s'^2_p}{n' p'^3} (x' x'_p + y' y'_p) \\ \Delta y' &= S \cdot \frac{y'_p s'^4}{2n' p'^3} (x'^2_p + y'^2_p) + P \cdot \frac{y'_p s'^2 s'^2_p}{2n' p'^3} (x'^2 + y'^2) - C \cdot \frac{y'_p s'^3 s'_p}{n' p'^3} (x' x'_p + y' y'_p) \\ &\quad + A \cdot \frac{y' s'^3 s'_p}{2n' p'^3} (x'^2_p + y'^2_p) + D \cdot \frac{y' s' s'^3_p}{2n' p'^3} (x'^2 + y'^2) - C \cdot \frac{y' s'^2 s'^2_p}{n' p'^3} (x' x'_p + y' y'_p)\end{aligned}$$

- Due to the derivative, only 5 terms remain for the primary aberrations, the transverse aberrations are of 3rd order in the coordinates (sum of powers in  $x', y', s'_x, s'_y$ )
- For the special cases of  $s' \rightarrow \infty$   
 $s'_p \rightarrow \infty$ 
  - image in infinity
  - exit pupil in infinity
there are particular sets of formulas
- Possible variables: different nomenclatures  
 $x', y', x'_p, y'_p, s', s'_p, p', A, A', i, i'$

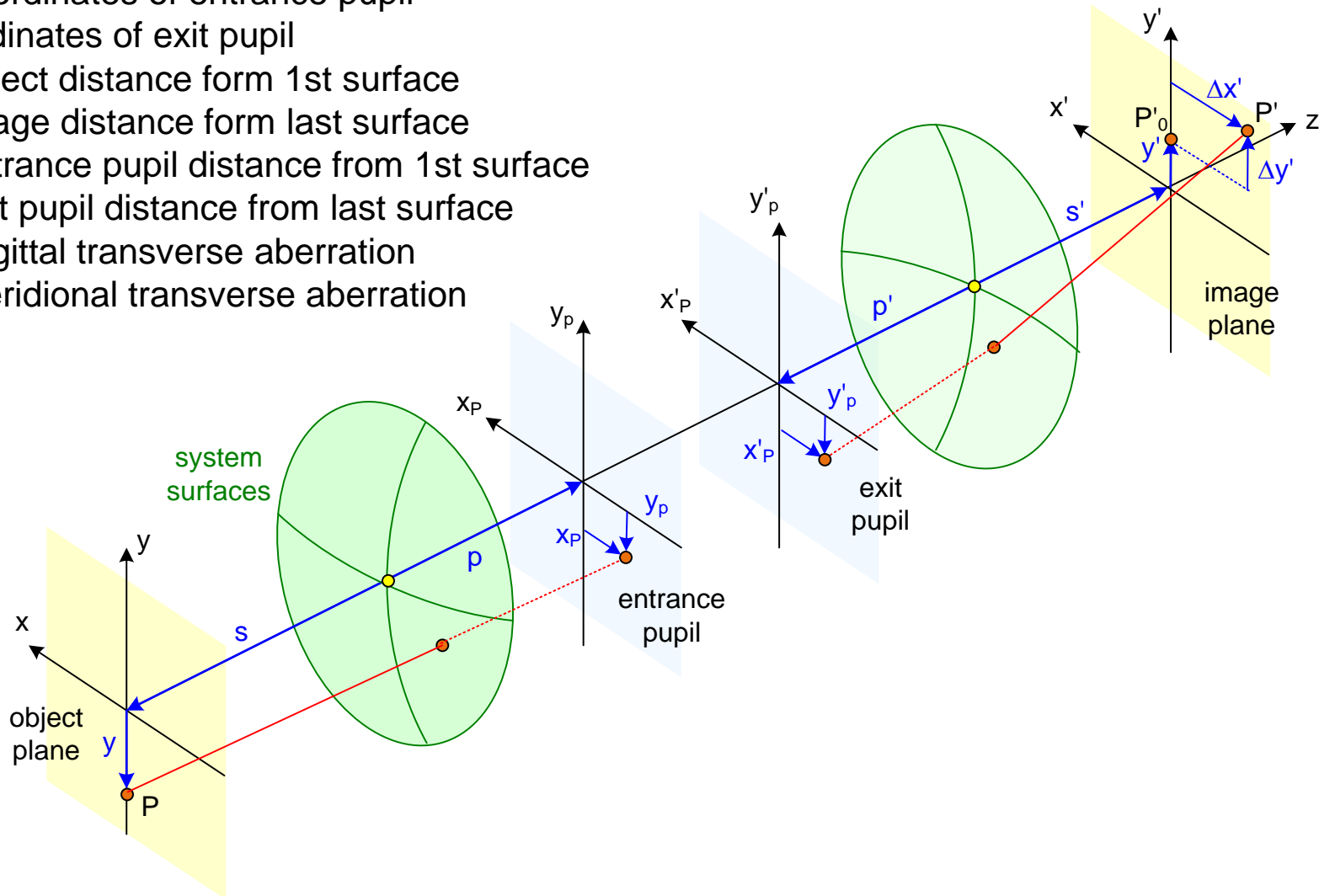
- Paraxial optics: small field and aperture angles, Aberrations occur for larger angle values
- Two-dimensional Taylor expansion shows field and aperture dependence
- Expansion for one meridional field point  $y$
- Pupil: cartesian or polar grid in  $x_p / y_p$





# Notations for an Optical System

$x, y$	object coordinates, especially object height
$x', y'$	image coordinates, especially image height
$x_p, y_p$	coordinates of entrance pupil
$x'_p, y'_p$	coordinates of exit pupil
$s$	object distance from 1st surface
$s'$	image distance from last surface
$p$	entrance pupil distance from 1st surface
$p'$	exit pupil distance from last surface
$\Delta x'$	sagittal transverse aberration
$\Delta y'$	meridional transverse aberration







# Sequence of Surfaces

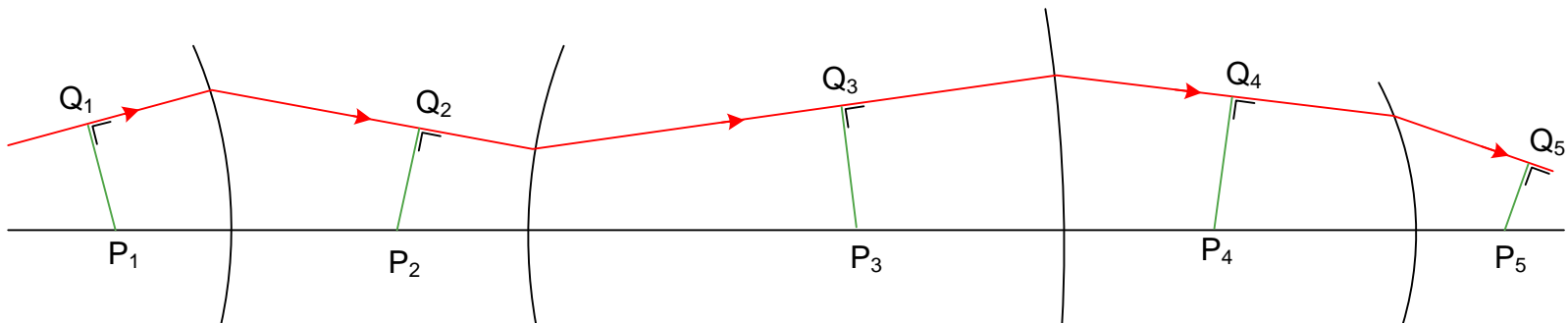
- Sequence of refracting surfaces:  
the optical path length contributions are additive
- The contributions of every surface are summed up
- Every surface contribution is imaged and magnified by the following surfaces

$$\Delta y' = \Delta y'_N + \sum_{k=1}^{N-1} m_{k+1} m_{k+2} \dots m_N \cdot \Delta y'_k, \quad \Delta x' = \Delta x'_N + \sum_{k=1}^{N-1} m_{k+1} m_{k+2} \dots m_N \cdot \Delta x'_k$$

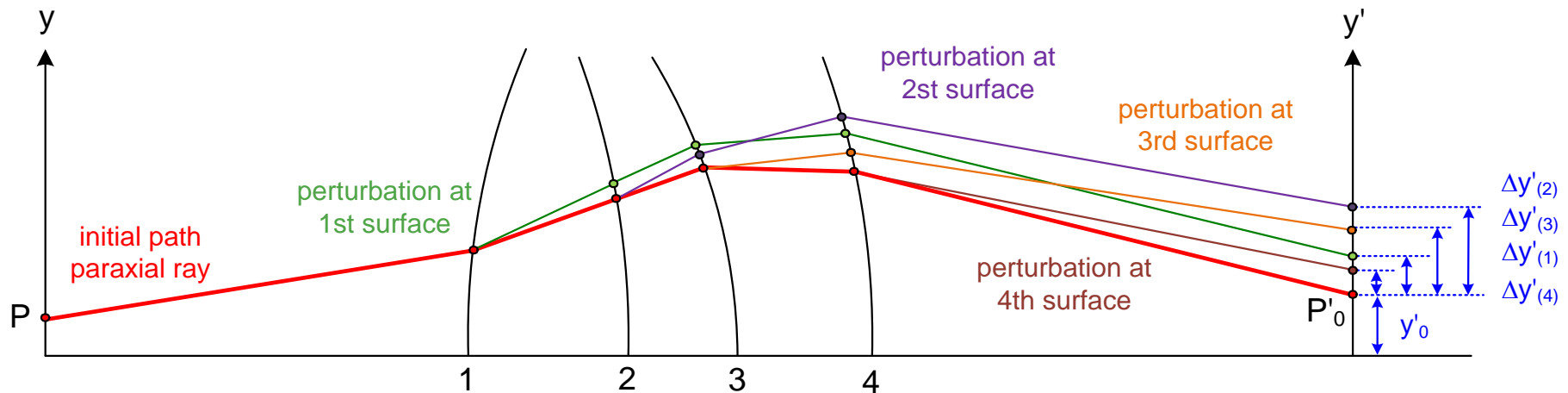
- The successive surfaces fulfill  $s_{j+1} = s'_j, \dots$
- The height ratios help to express the magnifications

$$\omega_j = \frac{h_j}{h_1} \quad \omega_{pj} = \frac{h_{pj}}{h_{p1}}$$

- Individual magnification dependencies for the various aberration types



- Special idea of Seidel to consider the 3rd order as a perturbation of the paraxial ray
- Independent changes/contributions of every surface aberration to the final transverse aberration
- Therefore special reference on paraxial fundamental properties





# Transverse Aberrations of Seidel

## ■ Decomposition of transverse aberrations

$$\begin{aligned}\Delta x' = & \frac{x'_p (x_p'^2 + y_p'^2) s'^4}{2n' p'^3} \cdot \sum_k S_k - \frac{[2x'_p (x'_p x'_p + y'_p y'_p) + x'_p (x_p'^2 + y_p'^2)] s'^3 s'_p}{2n' p'^3} \cdot \sum_k C_k \\ & + \frac{x'_p (x'_p x'_p + y'_p y'_p) s'^2 s_p'^2}{n' p'^3} \cdot \sum_k A_k + \frac{x'_p (x_p'^2 + y_p'^2) s'^2 s_p'^2}{2n' p'^3} \cdot \sum_k P_k \\ & - \frac{x'_p (x_p'^2 + y_p'^2) s' s_p'^3}{2n' p'^3} \cdot \sum_k D_k\end{aligned}$$

$$\begin{aligned}\Delta y' = & \frac{y'_p (x_p'^2 + y_p'^2) s'^4}{2n' p'^3} \cdot \sum_k S_k - \frac{[2y'_p (x'_p x'_p + y'_p y'_p) + y'_p (x_p'^2 + y_p'^2)] s'^3 s'_p}{2n' p'^3} \cdot \sum_k C_k \\ & + \frac{y'_p (x'_p x'_p + y'_p y'_p) s'^2 s_p'^2}{n' p'^3} \cdot \sum_k A_k + \frac{y'_p (x_p'^2 + y_p'^2) s'^2 s_p'^2}{2n' p'^3} \cdot \sum_k P_k \\ & - \frac{y'_p (x_p'^2 + y_p'^2) s' s_p'^3}{2n' p'^3} \cdot \sum_k D_k\end{aligned}$$



# Surface Contributions

- Spherical aberration

$$S_j = \omega_j^4 Q_j^2 \left( \frac{1}{n'_j s'_j} - \frac{1}{n_j s_j} \right)$$

- Coma

$$C_j = \omega_j^4 Q_j^2 \left( \frac{1}{n'_j s'_j} - \frac{1}{n_j s_j} \right) \cdot \frac{n_1 \omega_{pj} Q_{pj}}{\omega_j Q_j} \cdot \left( \frac{1}{s_1} - \frac{1}{s_{p1}} \right)$$

- Astigmatisms

$$A_j = \omega_j^4 Q_j^2 \left( \frac{1}{n'_j s'_j} - \frac{1}{n_j s_j} \right) \cdot \left( \frac{n_1 \omega_{pj} Q_{pj}}{\omega_j Q_j} \right)^2 \cdot \left( \frac{1}{s_1} - \frac{1}{s_{p1}} \right)^2$$

- Field curvature

$$P_j = \omega_j^4 Q_j^2 \left( \frac{1}{n'_j s'_j} - \frac{1}{n_j s_j} \right) \cdot \left( \frac{n_1 \omega_{pj} Q_{pj}}{\omega_j Q_j} \right)^2 \cdot \left( \frac{1}{s_1} - \frac{1}{s_{p1}} \right)^2 - \frac{1}{r_r} \cdot \left( \frac{1}{n'_j} - \frac{1}{n_j} \right)$$

- Distortion

$$D_j = \left[ \omega_j^4 Q_j^2 \left( \frac{1}{n'_j s'_j} - \frac{1}{n_j s_j} \right) \cdot \left( \frac{n_1 \omega_{pj} Q_{pj}}{\omega_j Q_j} \right)^2 \cdot \left( \frac{1}{s_1} - \frac{1}{s_{p1}} \right)^2 - \frac{1}{r_r} \cdot \left( \frac{1}{n'_j} - \frac{1}{n_j} \right) \right] \cdot \frac{n_1 \omega_{pj} Q_{pj}}{\omega_j Q_j} \left( \frac{1}{s_1} - \frac{1}{s_{p1}} \right)$$



# Circular Coordinates

- Introduction of circular coordinates according to the geometry

$$r'_p = \sqrt{x'^2_p + y'^2_p} \quad x'_p = r'_p \cdot \sin \varphi'_p$$

$$y_p = r'_p \cdot \cos \varphi'_p$$

- Transverse aberrations

$$\Delta x' = S \cdot \frac{x'_p s'^4}{2n' p'^3} r'^3_p \sin \varphi'_p + P \cdot \frac{s'^2 s'^2_p}{2n' p'^3} r'_p \sin \varphi'_p (x'^2 + y'^2) - C \cdot \frac{s'^3 s'_p}{n' p'^3} r'^2_p (x' \sin^2 \varphi'_p + y' \sin \varphi'_p \cos \varphi'_p)$$

$$+ A \cdot \frac{x' s'^3 s'_p}{2n' p'^3} r'^2_p + D \cdot \frac{x' s' s'^3_p}{2n' p'^3} (x'^2 + y'^2) - C \cdot \frac{x' s'^2 s'^2_p}{n' p'^3} r'_p (x' \sin \varphi'_p + y' \cos \varphi'_p)$$

$$\Delta y' = S \cdot \frac{s'^4}{2n' p'^3} r'^3_p \cos \varphi'_p + P \cdot \frac{s'^2 s'^2_p}{2n' p'^3} r'_p \cos \varphi'_p (x'^2 + y'^2) - C \cdot \frac{s'^3 s'_p}{n' p'^3} r'^2_p (x' \sin \varphi'_p \cos \varphi'_p + y' \cos^2 \varphi'_p)$$

$$+ A \cdot \frac{y' s'^3 s'_p}{2n' p'^3} r'^2_p + D \cdot \frac{y' s' s'^3_p}{2n' p'^3} (x'^2 + y'^2) - C \cdot \frac{y' s'^2 s'^2_p}{n' p'^3} r'_p (x' \sin \varphi'_p + y' \cos \varphi'_p)$$

- Aberration curves:
- Considering a ring with constant  $r'_p$  in the pupil,
  - eliminating  $\theta'_p$
  - delivers curve in image plane for  $\Delta x'$ ,  $\Delta y'$



# Simplified Formulas

- Simplified set of formulas:
  - field point only in  $y'$  considered
  - all system constants hidden in modified coefficients

$$\Delta y' = S' \cdot r_p'^3 \cos \varphi_p + C' \cdot y' \cdot r_p'^2 (2 + \cos 2\varphi_p) + (2A' + P') \cdot y'^2 \cdot r_p' \cos \varphi_p + D' \cdot y'^3$$

$$\Delta x' = S' \cdot r_p'^3 \sin \varphi_p + C' \cdot y' \cdot r_p'^2 \cdot \sin 2\varphi_p + P' \cdot y'^2 \cdot r_p' \sin \varphi_p$$


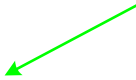

- Simple discussion of aberrations types:
  1. spherical aberration  $S'$ : no field dependence
  2. coma  $C'$ : grows linear with field
  3. astigmatism  $A'$ : double azimuthal angle dependence
  4. field curvature  $P'$ : interrelated with astigmatism
  5. distortion  $D'$ : no dependence on aperture, no blur
- Special case astigmatism and field curvature:
  - can alternatively be considered as tangential and sagittal image shell



# Polynomial Expansion of Aberrations

- Representation of 2-dimensional Taylor series vs field  $y$  and aperture  $r$
- Selection rules: checkerboard filling of the matrix
- Constant sum of exponents according to the order

		Field $y$ →						
			Spherical $y^0$	Coma $y^1$	Astigmatism $y^2$	$y^3$	$y^4$	$y^5$
Aper- ture $r$ ↓	Distortion	$r^0$		$y \cos\theta$ Tilt		$y^3 \cos\theta$ Distortion primary		$y^5 \cos\theta$ Distortion secondary
		$r^1$	$r^1$ Defocus		$y^2 r^1 \cos^2\theta$ $y^2 r^1$ Astig./Curvat.		$y^4 r^1 \cos^2\theta$ $y^4 r^1$	
		$r^2$		$y r^2 \cos\theta$ Coma primary		$y^3 r^2 \cos^3\theta$ $y^3 r^2 \cos\theta$		
		$r^3$	$r^3$ Spherical primary		$y^2 r^3 \cos^2\theta$ $y^2 r^3$			
		$r^4$		$y r^4 \cos\theta$ Coma secondary				
		$r^5$	$r^5$ Spherical secondary					

 **Image location**
 **Primary aberrations / Seidel**
 **Secondary aberrations**



# Power Expansion of Aberrations

- Orders of field and aperture dependencies for different representations of primary aberrations

Aberration	Coefficient	Seidel sum	Wave aberration		Transverse aberration		Longitudinal aberration	
			Aperture	Field	Aperture	Field	Aperture	Field
Spherical aberr.	$c_1$	$S_I$	4	0	3	0	2	0
Coma	$c_2$	$S_{II}$	3	1	2	1	1	1
Astigmatism	$c_3$	$S_{III}$	2	2	1	2	0	2
Field curvatures	(sagittal) $c_4$	(Petzval) $S_{IV}$	2	2	1	2	0	2
Distortion	$c_5$	$S_V$	1	3	0	3	-	-
Axial color	$\tilde{b}_1$	$C_I$	2	0	1	0	0	0
Lateral color	$\tilde{b}_2$	$C_{II}$	1	1	0	1	-	-



# Rotational Invariants

- General case : two coordinates in object plane and pupil
- Rotational symmetry: 3 invariants
  1. Scalar product of field vector and pupil vector

$$w = \vec{P} \cdot \vec{F} = P \cdot F \cdot \cos(\varphi_F - \varphi_P) = x_p \cdot x + y_p \cdot y$$

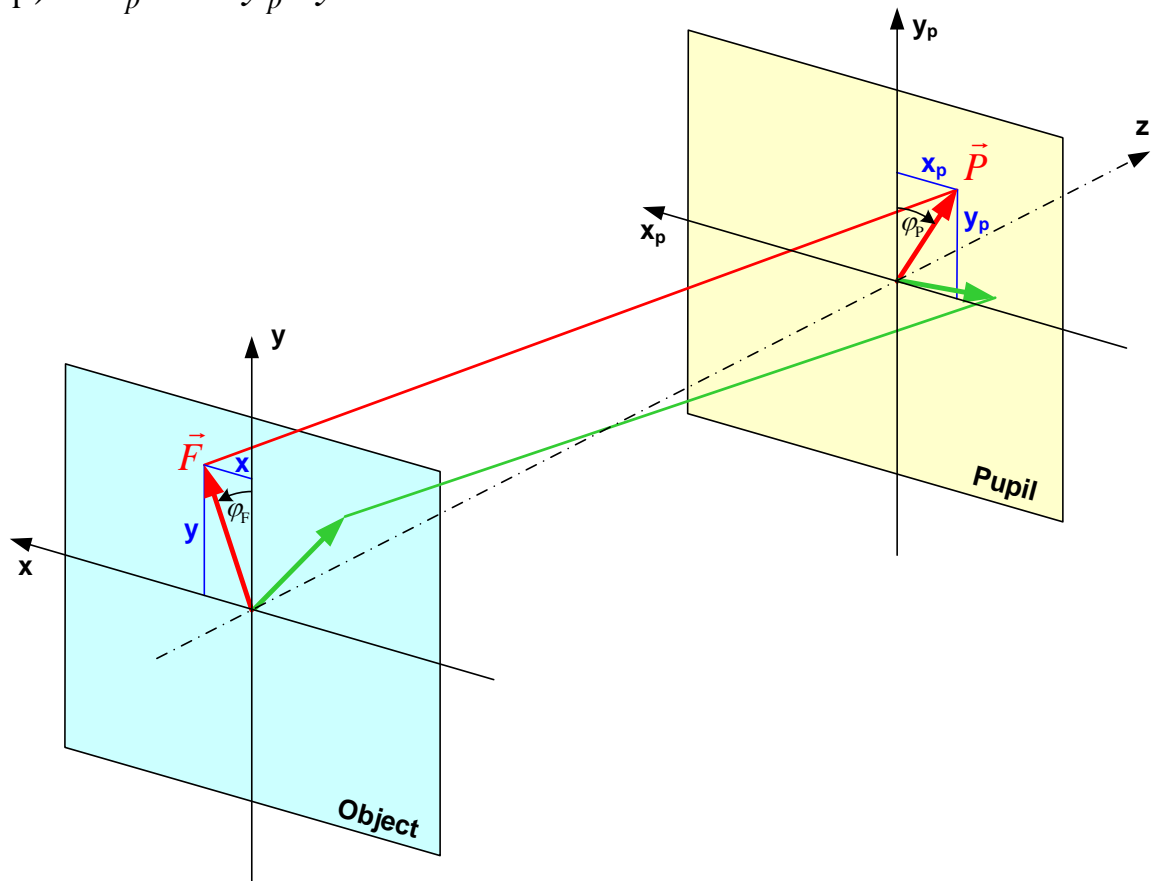
2. Square of field height

$$u = \vec{F} \cdot \vec{F} = F^2 = x^2 + y^2$$

3. Square of pupil height

$$v = \vec{P} \cdot \vec{P} = P^2 = x_p^2 + y_p^2$$

- Therefore:  
Only special power combinations are physically meaningful





# Power Series Expansion of Aberrations

- General case of Taylor expansion

$$W = \sum_{k,l,m,n} a_{klmn} x_p^k y_p^l x_p^m y_p^n$$

- Expansion with selection rules:  
only powers of the rotational invariants can occur

$$W = W(u, v, w)$$

- Simple expansion according to this scheme

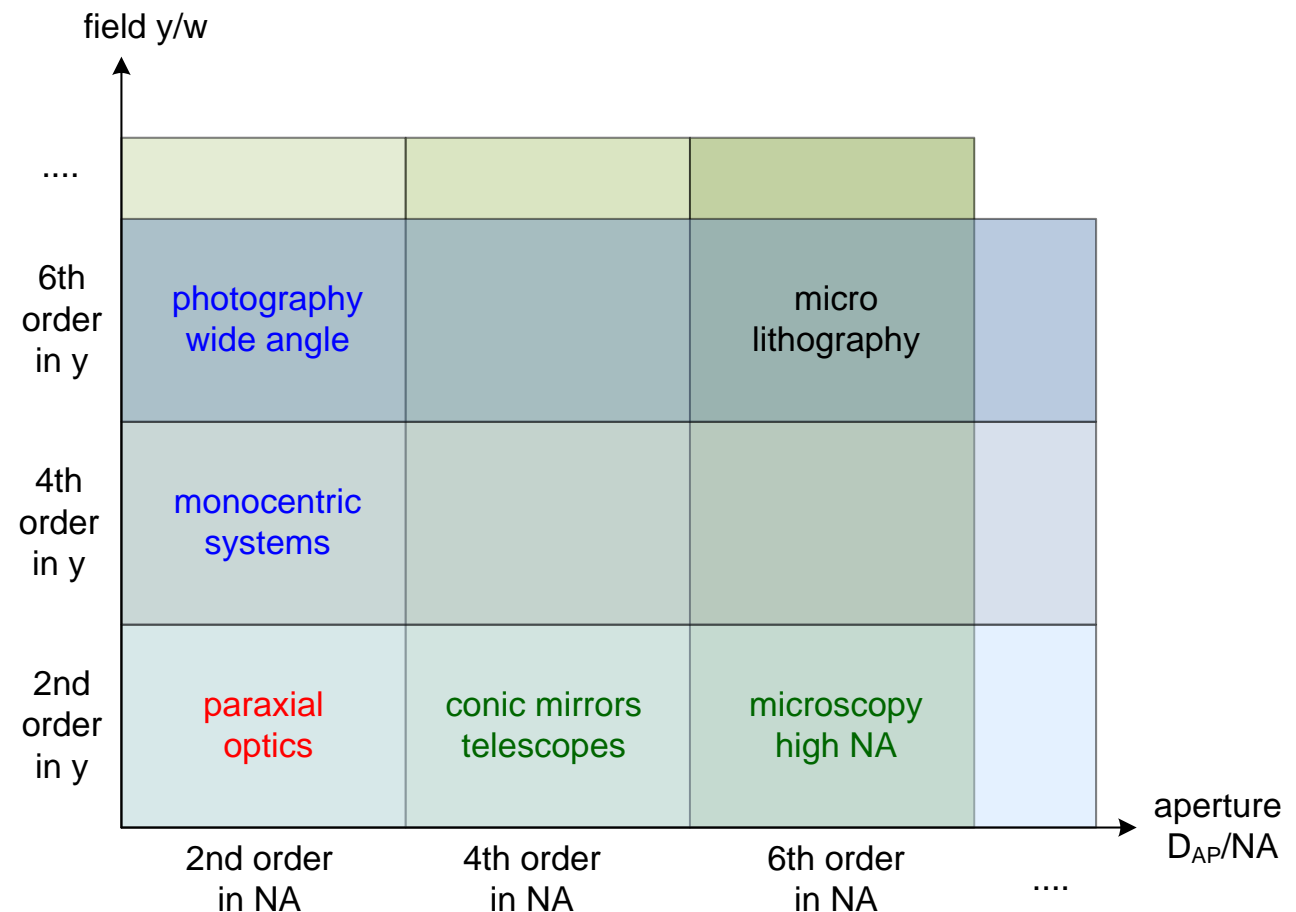
$$\begin{aligned} W = & a_0 + b_1 v + b_2 w + b_3 u \\ & + c_1 v^2 + c_2 wv + c_3 w^2 + c_4 uv + c_5 uw + c_6 u^2 \\ & + d_1 v^3 + d_2 wv^2 + d_3 w^2 v + d_4 uv^2 + d_5 uwv + d_6 w^3 + d_7 uw^2 + d_8 u^2 v + d_9 u^2 w + d_{10} u^3 + \dots \end{aligned}$$

- Explicite equation in real coordinates

$$\begin{aligned} W = & a_0 + b_1 (x_p^2 + y_p^2) + b_2 xy_p + b_3 y^2 \\ & + c_1 (x_p^2 + y_p^2)^2 + c_2 yy_p (x_p^2 + y_p^2) + c_3 y^2 y_p^2 + c_4 y^2 (x_p^2 + y_p^2) + c_5 y^3 y_p + c_6 y^4 \\ & + d_1 (x_p^2 + y_p^2)^3 + d_2 yy_p (x_p^2 + y_p^2)^2 + d_3 y^2 y_p^2 (x_p^2 + y_p^2) + d_4 y^2 (x_p^2 + y_p^2)^2 + d_5 y^3 y_p (x_p^2 + y_p^2) \\ & + d_6 y^3 y_p^3 + d_7 y^4 y_p^2 + d_8 y^4 (x_p^2 + y_p^2) + d_9 y^5 y_p + d_{10} y^6 + \dots \end{aligned}$$



- Relevance of higher order expansion terms
- Nearly perfect geometrical imaging possible in the special cases:
  1. small aperture
  2. small field





# 5<sup>th</sup> Order Aberrations

No	k Field power	l, Pupil power	m, Azimuthal power	Term	Name
1	0	6	0	$W_{060} r_p^6$	Secondary spherical aberration
2	1	5	1	$W_{151} y' r_p^5 \cos \theta$	Secondary coma
3	2	4	2	$W_{242} y'^2 r_p^4 \cos^2 \theta$	Secondary astigmatism, wing error
4	3	3	3	$W_{333} y'^3 r_p^3 \cos^3 \theta$	trefoil error, arrow error
5	2	4	0	$W_{240} y'^2 r_p^4$	Skew spherical aberration
6	3	3	1	$W_{331} y'^3 r_p^3 \cos \theta$	Skew coma
7	4	2	2	$W_{422} y'^4 r_p^2 \cos^2 \theta$	Skew astigmatism
8	4	2	0	$W_{420} y'^4 r_p^2$	Secondary field curvature
9	5	1	1	$W_{511} y'^5 r_p \cos \theta$	Secondary distortion



# Notations and General View

- There is a large number of different notations for the third order representation:  
Haferkorn, Welford, Seidel, Berek, Köhler,...
- The differences are the choice of the parameter and some of the approximations
- The so called reduced representations are of a special form without considering the field dependence explicite  
Example: Zernike expansion
- The third order theory is limited on systems with not too high angles of marginal and chief ray
- Mostly the third order describes the leading term
- Higher orders than 3 can not be decomposed as simple into the surface contributions:
  1. the magnification of the following surfaces acts non-linear and can not be neglected
  2. the second order perturbation theory has no decoupling of the contributions  
(induced aberrations)
- Also available: analytical aberration coefficients for gradient media (F. Bociort)



# Welford's Notation

- Abbreviations

bar: chief ray

$$A = n(hc + u) = n \cdot i = n' \cdot i'$$

$$\bar{A} = n(\bar{h}c + \bar{u}) = n \cdot \bar{i} = n' \cdot \bar{i}'$$

- Seidel aberrations

1. spherical aberration

$$S_I = -\sum A^2 \cdot h \cdot \Delta \left( \frac{u}{n} \right)$$

2. coma

$$S_{II} = -\sum \bar{A}A \cdot h \cdot \Delta \left( \frac{u}{n} \right)$$

3. astigmatism

$$S_{III} = -\sum \bar{A}^2 \cdot h \cdot \Delta \left( \frac{u}{n} \right)$$

4. Petzval curvature

$$S_{IV} = -\sum H^2 \cdot c \cdot \Delta \left( \frac{1}{n} \right)$$

5. distortion

$$S_V = -\sum \left\{ \frac{\bar{A}^3}{A} \cdot h \cdot \Delta \left( \frac{u}{n} \right) + \frac{\bar{A}}{A} \cdot H^2 \cdot c \cdot \Delta \left( \frac{1}{n} \right) \right\}$$

- Representation in normalized circular coordinates

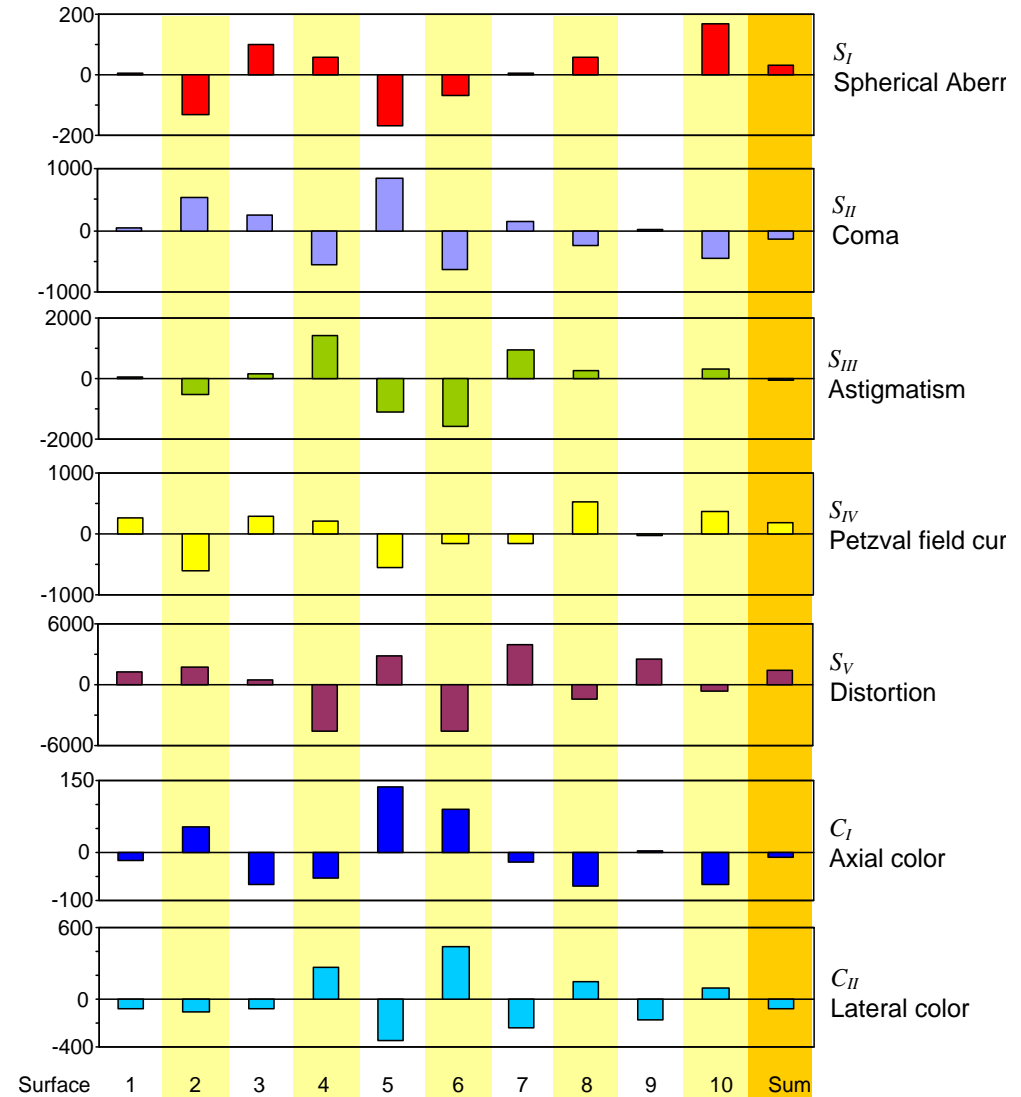
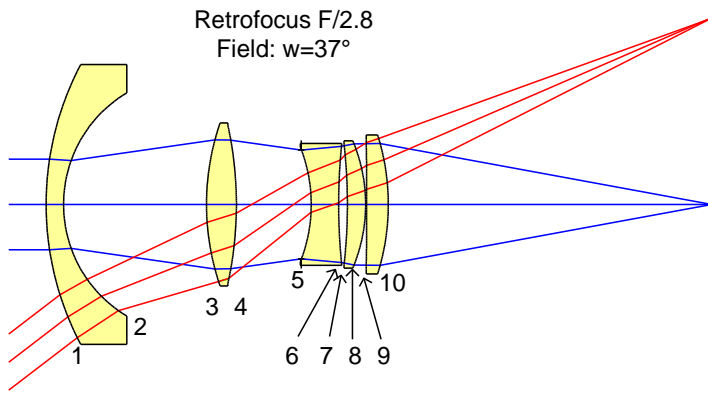
$$\rho = \frac{r}{r_{\max}} \quad , \quad \eta = \frac{y}{y_{\max}}$$

$$W(\rho, \varphi, \eta) = \frac{1}{8} S_I \rho^4 + \frac{1}{2} S_{II} \eta \rho^3 \cos \varphi + \frac{1}{2} S_{III} \eta^2 \rho^2 \cos^2 \varphi + \frac{1}{4} (S_{III} + S_{IV}) \eta^2 \rho^2 + \frac{1}{2} S_V \eta^3 \rho \cos \varphi$$



# Surface Contributions: Example

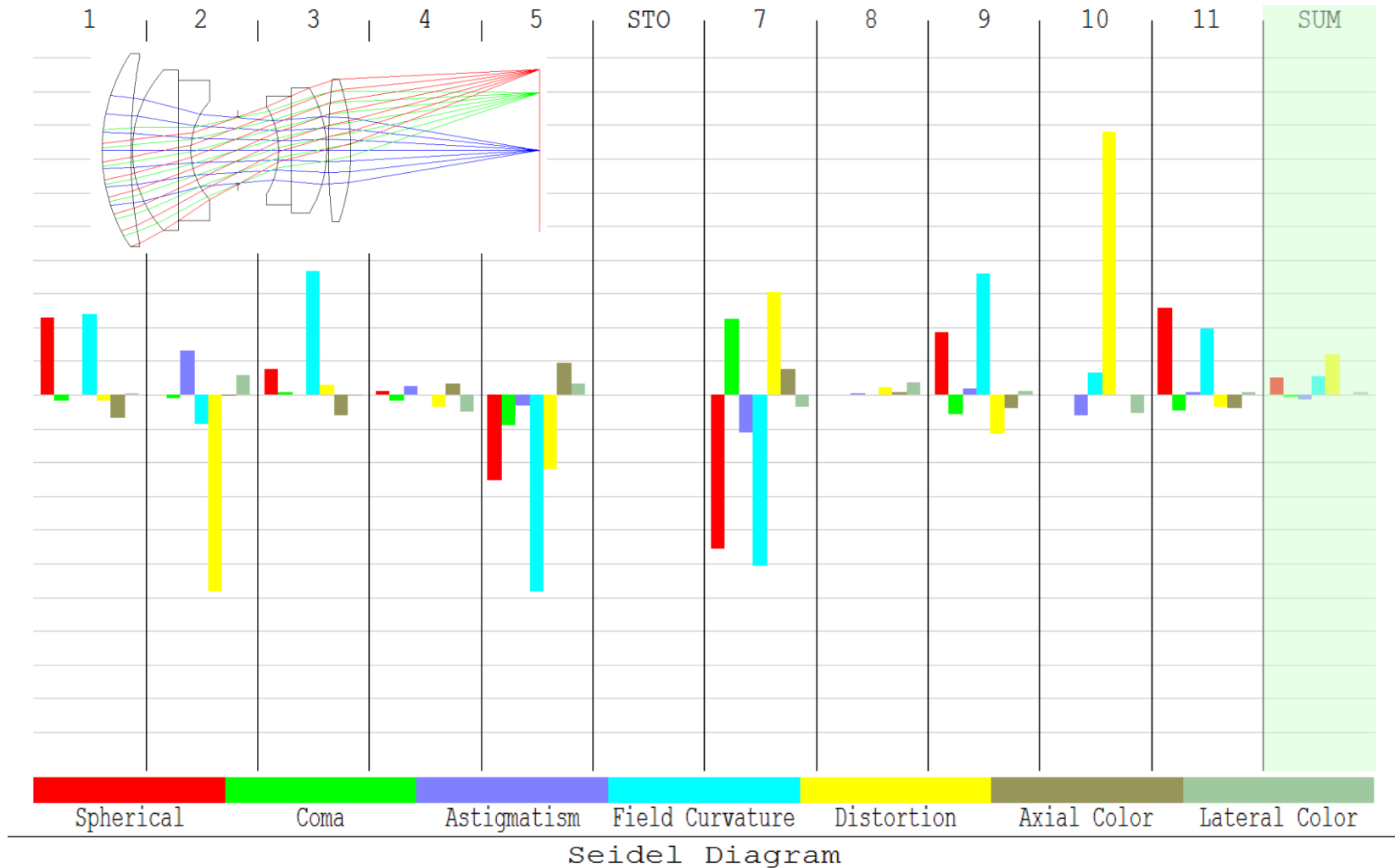
- Seidel aberrations: representation as sum of surface contributions possible
- Gives information on correction of a system
- Example: photographic lens





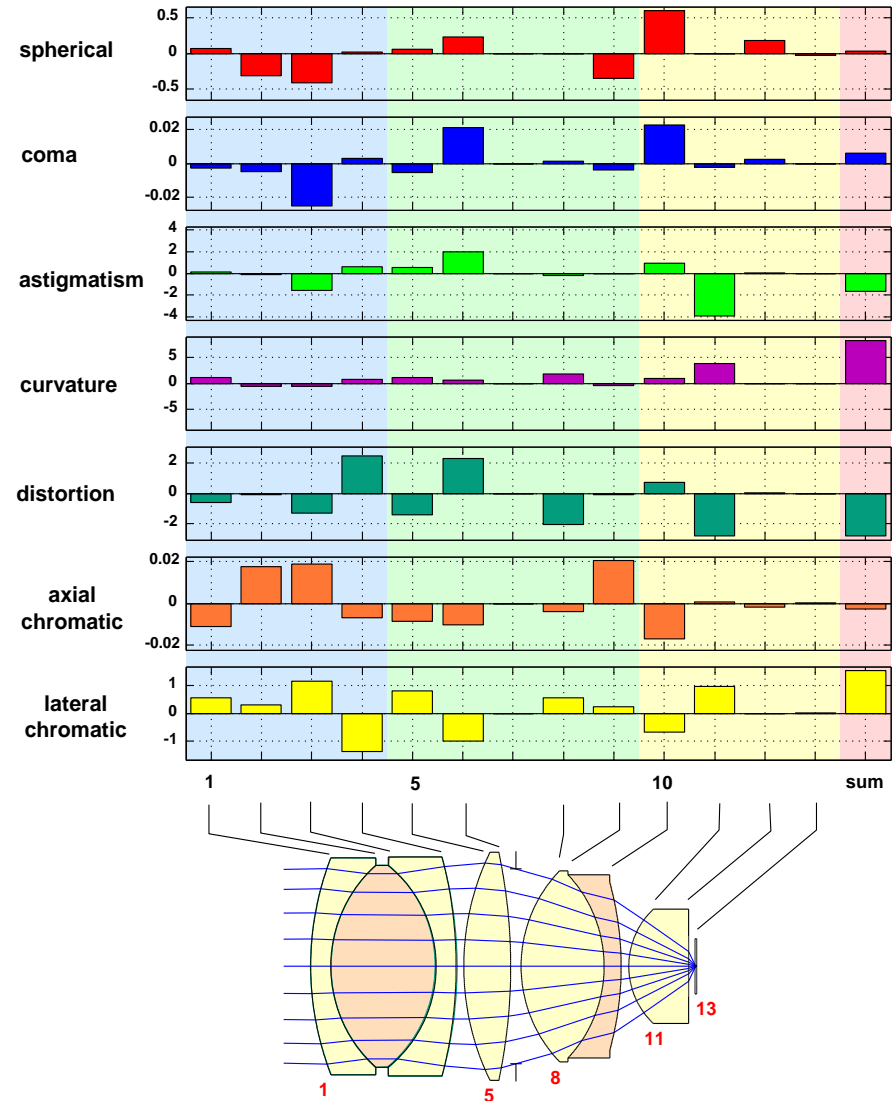
# Seidel Surface Contributions

- Graphical supported representation of the Seidel surface contributions of a photographic lens

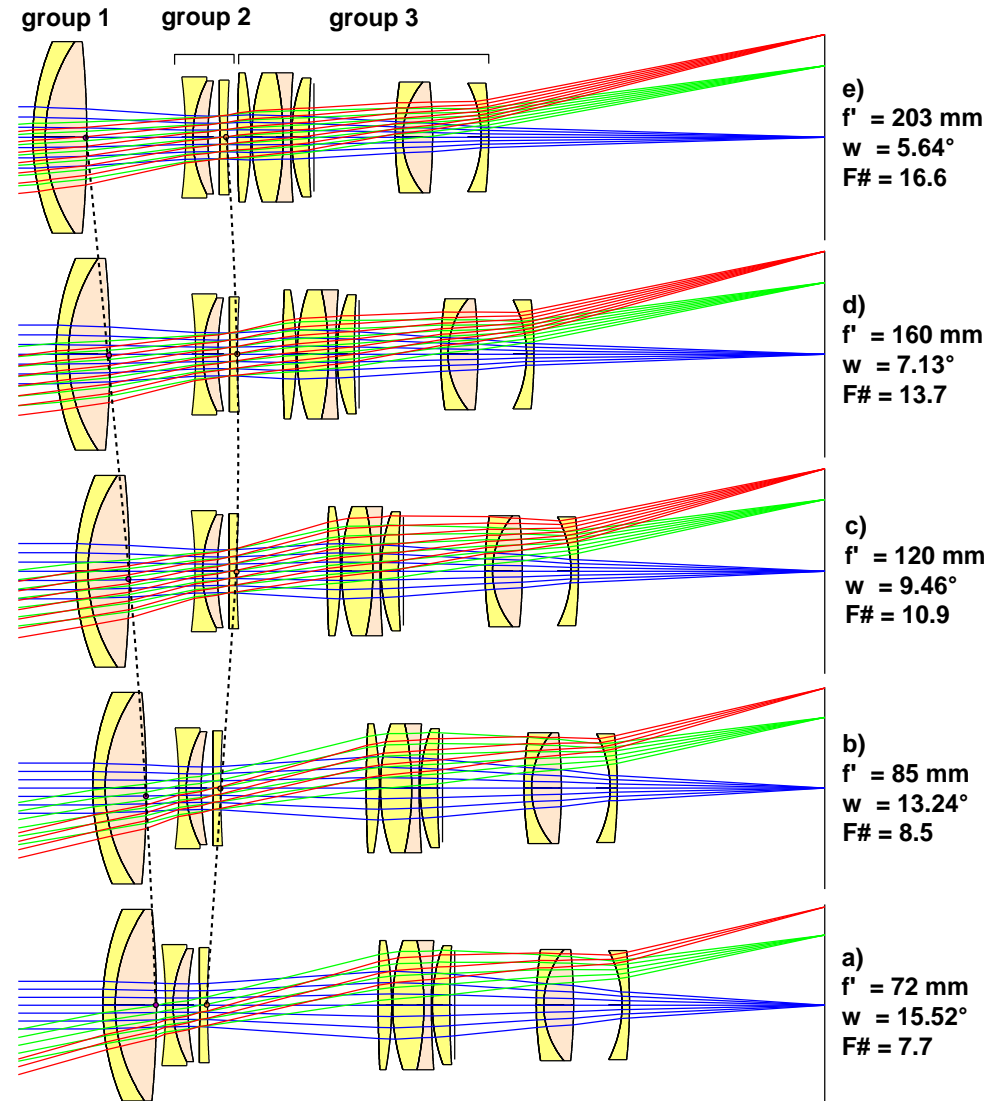




- Seidel surface contributions for 100x/0.90
- No field flattening group
- Lateral color in tube lens corrected



- Zoom lens
- Three moving groups



# Performance Variation over z

Seidel  
surface  
contrib.

